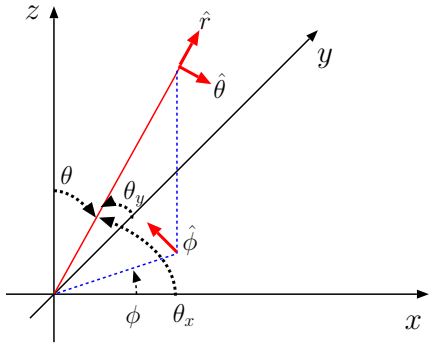
Direction cosines and polar angles:



Direction cosines

$$\cos\theta_x = \sin\theta\cos\phi, \quad \cos\theta_y = \sin\theta\sin\phi,$$

imply that

$$\sin\theta_x = \sqrt{1-\sin^2\theta\cos^2\phi}, \quad \sin\theta_y = \sqrt{1-\sin^2\theta\sin^2\phi},$$

and

$$\cos^2\theta_x + \cos^2\theta_y + \cos^2\theta = 1,$$

and all *polar angles* θ_x, θ_y, and $\theta_z \equiv \theta$ are considered in 0 to π interval only. The corresponding unit vectors in the direction of increasing θ_x, θ_y, and θ obey

$$\hat{\theta}_x \equiv \frac{\hat{r}\times\hat{r}\times\hat{x}}{\sin\theta_x} = \frac{\hat{r}(\hat{r}\cdot\hat{x})-\hat{x}}{\sin\theta_x} = \frac{\hat{r}\cos\theta_x-\hat{x}}{\sin\theta_x}, \;\; \hat{\theta}_y = \frac{\hat{r}\cos\theta_y-\hat{y}}{\sin\theta_y}, \;\; \hat{\theta} = \frac{\hat{r}\cos\theta-\hat{z}}{\sin\theta}.$$

Since

$$\hat{\phi} = \hat{y}\cos\phi - \hat{x}\sin\phi$$

it follows that:

- $\hat{\theta}_x\cdot\hat{\theta}_y = \frac{\hat{r}\cos\theta_x-\hat{x}}{\sin\theta_x}\cdot\frac{\hat{r}\cos\theta_y-\hat{y}}{\sin\theta_y} = -\frac{\cos\theta_x\cos\theta_y}{\sin\theta_x\sin\theta_y}$,
- $\hat{\theta}_x\cdot\hat{\theta} = \frac{\hat{r}\cos\theta_x-\hat{x}}{\sin\theta_x}\cdot\frac{\hat{r}\cos\theta-\hat{z}}{\sin\theta} = -\frac{\cos\theta_x\cos\theta}{\sin\theta_x\sin\theta}$,
- $\hat{\theta}_y\cdot\hat{\theta} = \frac{\hat{r}\cos\theta_y-\hat{y}}{\sin\theta_y}\cdot\frac{\hat{r}\cos\theta-\hat{z}}{\sin\theta} = -\frac{\cos\theta_y\cos\theta}{\sin\theta_y\sin\theta}$,
- $\hat{\theta}_x\cdot\hat{\phi} = \frac{\hat{r}\cos\theta_x-\hat{x}}{\sin\theta_x}\cdot\hat{\phi} = -\frac{\hat{x}\cdot\hat{\phi}}{\sin\theta_x} = -\frac{\hat{x}\cdot(\hat{y}\cos\phi-\hat{x}\sin\phi)}{\sin\theta_x} = \frac{\sin\phi}{\sin\theta_x}$,
- $\hat{\theta}_y\cdot\hat{\phi} = \frac{\hat{r}\cos\theta_y-\hat{y}}{\sin\theta_y}\cdot\hat{\phi} = -\frac{\hat{y}\cdot\hat{\phi}}{\sin\theta_y} = -\frac{\hat{y}\cdot(\hat{y}\cos\phi-\hat{x}\sin\phi)}{\sin\theta_y} = -\frac{\cos\phi}{\sin\theta_y}$,
- $\hat{\theta}_x\cdot\hat{r} = \frac{\hat{r}\cos\theta_x-\hat{x}}{\sin\theta_x}\cdot\hat{r} = 0$,
- $\hat{\theta}_y\cdot\hat{r} = \frac{\hat{r}\cos\theta_y-\hat{y}}{\sin\theta_y}\cdot\hat{r} = 0$.