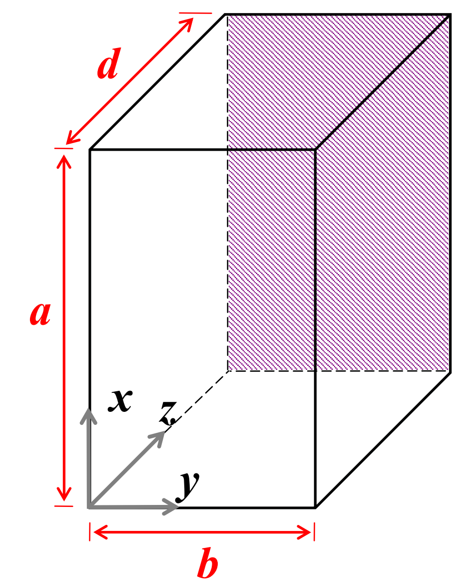
33 Rectangular cavities



- Consider a rectangular waveguide propagating some TE$_{mn}$ mode having a longitudinal magnetic field component

$$H_z^+ \propto \cos(k_x x)\cos(k_y y)e^{-jk_z z},$$

where

$$k_x = \frac{m\pi}{a}, \quad k_y = \frac{n\pi}{b}, \quad \text{and} \quad k_z = \sqrt{k^2 - k_x^2 - k_y^2}.$$

In principle, the same guide can also propagate a TE$_{mn}$ mode field with

$$H_z^- \propto \cos(k_x x)\cos(k_y y)e^{+jk_z z}$$

in reverse direction, and if both waves were present in the guide, we would have a total field

$$H_{zt} = \cos(k_x x)\cos(k_y y)(Fe^{-jk_z z} + Re^{+jk_z z})$$

where F and R denote the amplitudes of the forward and reverse waves depending on sources and/or boundaries in z.

  - Of course $E_{zt} = 0$ for TE$_{mn}$ modes, while
  - transverse field components can be obtained using the equations shown in the margin (derived in Lecture 29) where the sign of $\mp jk_z$ is taken in accordance with the order implied in $e^{\mp jk_z z}$ for forward and reverse propagating waves.

TE mode fields:

$$H_x^\pm = \frac{\mp jk_z \frac{\partial H_z}{\partial x}}{k^2 - k_z^2},$$

$$H_y^\pm = \frac{\mp jk_z \frac{\partial H_z}{\partial y}}{k^2 - k_z^2},$$

$$E_y^\pm = \frac{j\omega\mu_o \frac{\partial H_z}{\partial x}}{k^2 - k_z^2},$$

$$E_x^\pm = \frac{-j\omega\mu_o \frac{\partial H_z}{\partial y}}{k^2 - k_z^2}.$$

- For a **rectangular cavity** formed by introducing conducting walls at $z = 0$ and $z = d$ within a rectangular waveguide, the pertinent boundary conditions to be imposed on H_{zt} become

$$H_{zt}(x, y, 0) = 0 \text{ and } H_{zt}(x, y, d) = 0$$

since $\mathbf{H}$ cannot be perpendicular to a conducting plate. Accordingly,

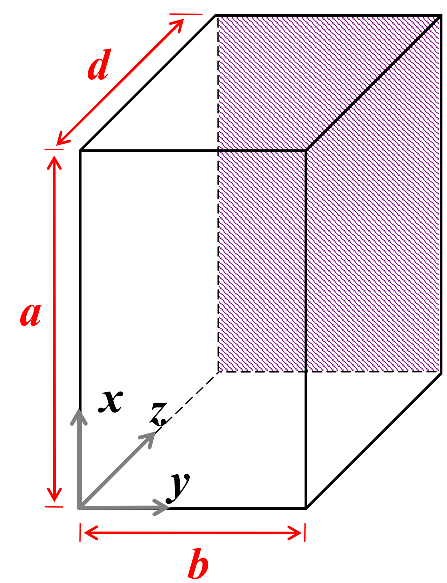
1. $H_{zt}(x, y, 0) = 0$ requires $R = -F$, in which case we can write (taking F=1 for simplicity)

$$\begin{aligned} H_{zt} &= \cos(k_x x)\cos(k_y y)(e^{-jk_z z} - e^{+jk_z z}) \\ &= -j2\cos(k_x x)\cos(k_y y)\sin(k_z z). \end{aligned}$$

2. $H_{zt}(x, y, d) = 0$, in turn, requires

$$k_z d = l\pi, \quad l = 1, 2, 3 \cdots$$

excluding $l = 0$ for non-zero H_{zt}.



- $H_{zt}(x, y, z)$ now describes a standing wave pattern within the rectangular cavity, having a periodicity in z where

$$|H_{zt}(x, y, z)| \propto |\sin(k_z z)|$$

repeats over z by integer multiples of $\lambda_z/2$, where

$$\lambda_z = \frac{2\pi}{k_z} \quad \text{and} \quad k_z = \frac{l\pi}{d}, \quad l = 1, 2, 3 \cdots$$

- These standing waves are termed TE_{mnl} modes and oscillate with characteristic frequencies

$$f = \frac{\omega}{2\pi} = \sqrt{(\frac{mc}{2a})^2 + (\frac{nc}{2b})^2 + (\frac{lc}{2d})^2} = \frac{c}{2}\sqrt{(\frac{m}{a})^2 + (\frac{n}{b})^2 + (\frac{l}{d})^2} \equiv f_{mnl}$$

  that follow from

$$k_x = \frac{m\pi}{a}, \quad k_y = \frac{n\pi}{b}, \quad k_z = \frac{l\pi}{d}$$

  implying

$$k^2 = \frac{\omega^2}{c^2} = k_x^2 + k_y^2 + k_z^2 = (\frac{m\pi}{a})^2 + (\frac{n\pi}{b})^2 + (\frac{l\pi}{d})^2.$$

Characteristic frequencies f_{nml} are also known as **resonance frequencies** of the cavity, since they represent a discrete set of frequencies for which source-free field variations are possible within the cavity, in analogy with

1. having source-free voltage variations in an ideal LC circuit at its resonance frequency $\omega = \frac{1}{\sqrt{LC}}$, and also in analogy with

2. TL resonators studied in ECE 329 (short or open circuited TL segments).

- Transverse field components of TE_{mnl} resonances can be obtained by superposing the transverse derivatives of

$$H_z^{\pm} = \pm\cos(k_x x)\cos(k_y y)e^{\mp jk_z z}$$

as specified in the margin. We that find

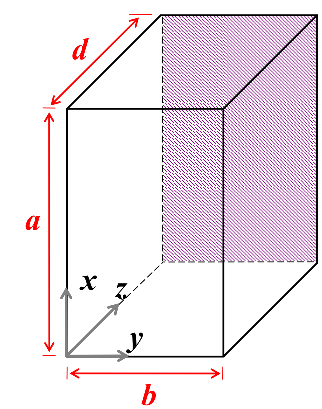
$$\begin{aligned} H_x^{\pm} &= \frac{jk_z k_x \sin(k_x x)\cos(k_y y)e^{\mp jk_z z}}{k^2 - k_z^2} \\ H_y^{\pm} &= \frac{jk_z k_y \cos(k_x x)\sin(k_y y)e^{\mp jk_z z}}{k^2 - k_z^2} \\ E_y^{\pm} &= \frac{\mp j\omega\mu_o k_x \sin(k_x x)\cos(k_y y)e^{\mp jk_z z}}{k^2 - k_z^2} \\ E_x^{\pm} &= \frac{\pm j\omega\mu_o k_y \cos(k_x x)\sin(k_y y)e^{\mp jk_z z}}{k^2 - k_z^2} \end{aligned}$$

which in turn lead to

$$\begin{aligned} H_{zt} &= H_z^+ + H_z^- = -j2\cos(k_x x)\cos(k_y y)\sin(k_z z) \\ H_{xt} &= H_x^+ + H_x^- = \frac{j2k_z k_x \sin(k_x x)\cos(k_y y)\cos(k_z z)}{k^2 - k_z^2} \\ H_{yt} &= H_y^+ + H_y^- = \frac{j2k_z k_y \cos(k_x x)\sin(k_y y)\cos(k_z z)}{k^2 - k_z^2} \\ E_{yt} &= E_y^+ + E_y^- = \frac{-2\omega\mu_o k_x \sin(k_x x)\cos(k_y y)\sin(k_z z)}{k^2 - k_z^2} \\ E_{xt} &= E_x^+ + E_x^- = \frac{2\omega\mu_o k_y \cos(k_x x)\sin(k_y y)\sin(k_z z)}{k^2 - k_z^2}. \end{aligned}$$

TE mode fields:

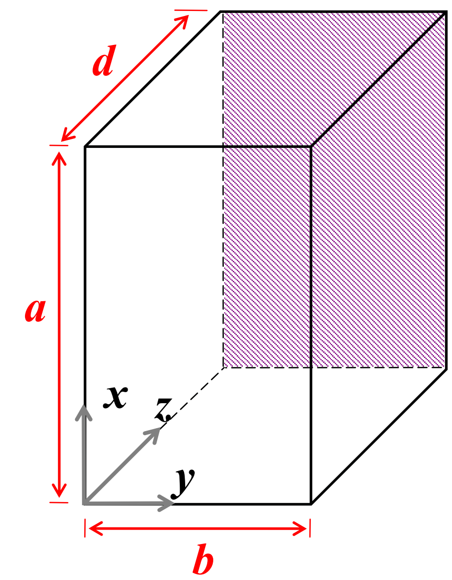
$$\begin{aligned} H_x^{\pm} &= \frac{\mp jk_z \frac{\partial H_z}{\partial x}}{k^2 - k_z^2}, \\ H_y^{\pm} &= \frac{\mp jk_z \frac{\partial H_z}{\partial y}}{k^2 - k_z^2}, \\ E_y^{\pm} &= \frac{j\omega\mu_o \frac{\partial H_z}{\partial x}}{k^2 - k_z^2}, \\ E_x^{\pm} &= \frac{-j\omega\mu_o \frac{\partial H_z}{\partial y}}{k^2 - k_z^2}. \end{aligned}$$



- Standing waves formed with superposed TM_{mn} mode fields having z-components

$$E_z^{\pm} = \sin(k_x x)\sin(k_y y)e^{\mp jk_z z}$$

will likewise produce TM_{mnl} mode resonances in rectangular cavities of dimensions $a > b$ and d having identical resonant frequencies as TE_{mnl} modes.



- For TM_{mnl} modes with $H_{zt} = 0$, a longitudinal standing wave field

$$E_{zt} = E_z^+ + E_z^- = \sin(k_x x)\sin(k_y y)(e^{-jk_z z} + e^{+jk_z z})$$

leads to transverse field components satisfying the boundary conditions at $z = 0$ and $z = d$ provided that

$$k_z = \frac{l\pi}{d}, \quad l = 0, 1, 2, 3 \cdots$$

  - $l = 0$ is allowed in this case since $k_z = 0$ does not lead to "incompatible" boundary conditions (normal E_z and tangential $H_{x,y}$ are allowed on conducting walls at $z = 0$ and d)
  - on the other hand, it is required that m and n are *both* non-zero, a property inherited from propagating TM_{mn} modes.

- TM_{mn} transverse field components accompanying

$$E_z^{\pm} = \sin(k_x x)\sin(k_y y)e^{\mp jk_z z}$$

can be found from the relations given in the margin. They lead to

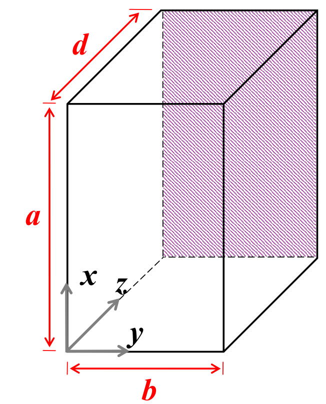
$$\begin{aligned} H_x^{\pm} &= \frac{j\omega\epsilon_o k_y \sin(k_x x)\cos(k_y y)e^{\mp jk_z z}}{k^2 - k_z^2} \\ H_y^{\pm} &= \frac{-j\omega\epsilon_o k_x \cos(k_x x)\sin(k_y y)e^{\mp jk_z z}}{k^2 - k_z^2} \\ E_y^{\pm} &= \frac{\mp jk_z k_y \sin(k_x x)\cos(k_y y)e^{\mp jk_z z}}{k^2 - k_z^2} \\ E_x^{\pm} &= \frac{\mp jk_z k_x \cos(k_x x)\sin(k_y y)e^{\mp jk_z z}}{k^2 - k_z^2} \end{aligned}$$

from which

$$\begin{aligned} E_{zt} &= E_z^+ + E_z^- = 2\sin(k_x x)\sin(k_y y)\cos(k_z z) \\ H_{xt} &= H_x^+ + H_x^- = \frac{j2\omega\epsilon_o k_y \sin(k_x x)\cos(k_y y)\cos(k_z z)}{k^2 - k_z^2} \\ H_{yt} &= H_y^+ + H_y^- = \frac{-j2\omega\epsilon_o k_x \cos(k_x x)\sin(k_y y)\cos(k_z z)}{k^2 - k_z^2} \\ E_{yt} &= E_y^+ + E_y^- = \frac{-2k_z k_y \sin(k_x x)\cos(k_y y)\sin(k_z z)}{k^2 - k_z^2} \\ E_{xt} &= E_x^+ + E_x^- = \frac{-2k_z k_x \cos(k_x x)\sin(k_y y)\sin(k_z z)}{k^2 - k_z^2}. \end{aligned}$$

TM mode fields:

$$\begin{aligned} H_x^{\pm} &= \frac{j\omega\epsilon_o \frac{\partial E_z}{\partial y}}{k^2 - k_z^2}, \\ H_y^{\pm} &= \frac{-j\omega\epsilon_o \frac{\partial E_z}{\partial x}}{k^2 - k_z^2}, \\ E_y^{\pm} &= \frac{\mp jk_z \frac{\partial E_z}{\partial y}}{k^2 - k_z^2}, \\ E_x^{\pm} &= \frac{\mp jk_z \frac{\partial E_z}{\partial x}}{k^2 - k_z^2}. \end{aligned}$$



- Notice that $E_{xt} = 0$ and $E_{yt} = 0$ at both $z = 0$ and $z = d$ provided that $k_z d = l\pi$, as claimed earlier on.
- Furthermore $l = 0$ does not lead to a trivial field since in that case E_{zt}, H_{xt}, and H_{yt} are non vanishing!

- Summarizing the results from above, in a rectangular cavity of dimensions $a > b$ and d and conducting walls, resonant field oscillations at distinct set of frequencies

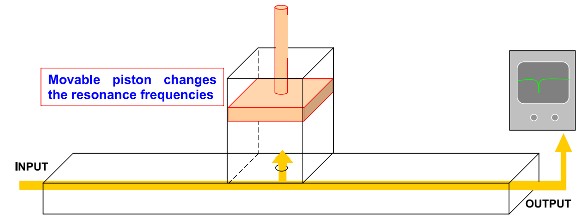
$$f_{mnl} = \frac{c}{2}\sqrt{(\frac{m}{a})^2 + (\frac{n}{b})^2 + (\frac{l}{d})^2}$$

are possible, so long as at least two of the indices m, n, and l are non zero.

  - For TE_{mnl} resonances $m = 0$ or $n = 0$ are permitted,
  - For TM_{mnl} resonances only $l = 0$ is permitted,
  - A resonance of frequency f_{mnl} is is said to be *non-degenerate* if it is allowed for a single mode and it is *degenerate* otherwise.

- **Practical uses of cavities:**

  1. Cavities with small apertures on their walls will interact strongly with external signals (suck them in) having oscillation frequencies matching one of the resonant frequencies, and, conversely, weakly at off-resonant external frequencies. This leads to the usage of cavities as "frequency meters".



  2. Dielectric filled cavities will have resonant frequencies

$$f_{mnl} = \frac{v_p}{2}\sqrt{(\frac{m}{a})^2 + (\frac{n}{b})^2 + (\frac{l}{d})^2} \text{ where } v_p = \frac{1}{\sqrt{\mu\epsilon}}.$$

     Measuring the resonant frequencies of a dielectric-filled cavity is a very accurate means of determining $\sqrt{\mu\epsilon}$.

  3. Cavities filled with active media or devices are a common way of configuring practical signal sources — e.g., lasers.

  4. Microwave ovens are essentially resonant cavities excited by (coupled to) a source operating near some of the resonant frequencies of the cavity that establishes a reasonably smooth field structure where the food is to be placed.

- Our analysis of cavities and waveguides have been based on the assumption of perfectly conducting walls, so far. Waveguide and cavity walls will in practice be very good but imperfect conductors. The implications of this are:

1. Propagating waveguide modes will be weakly attenuated as the field energy is lost into the walls to drive ohmic currents within a few skin-depths of the metallic surface.

   In our idealization of the walls as "perfect conductors", we refer to the depth integral of these volumetric current densities as "surface current densities". In general, an equivalent surface current $\tilde{\mathbf{J}}_s$ on a wall will deliver an average power of

$$S_{loss} = \frac{1}{2} R_s |\tilde{\mathbf{J}}_s|^2 = \frac{1}{2}\sqrt{\frac{\pi f \mu}{\sigma}} |\tilde{\mathbf{J}}_s|^2 \, \frac{\mathrm{W}}{\mathrm{m}^2}$$

   to be dissipated per unit area of the wall (see margin and the ECE 329 notes).

2. Cavity mode oscillations at frequencies f_{mnl} will be damped as a function of time if not "replenished". The rate of energy loss P_{loss} can be calculated by integrating

$$S_{loss} = \frac{1}{2}\sqrt{\frac{\pi f_{mnl} \mu}{\sigma}} |\tilde{\mathbf{J}}_s|^2 \, \frac{\mathrm{W}}{\mathrm{m}^2}$$

   over the 6 cavity walls where at each wall we use $|\tilde{\mathbf{J}}_s|^2 = |\tilde{\mathbf{H}}_{tangential}|^2$. Decay time-constant of the stored mode-energy W, the volume integral of $\frac{1}{4}\epsilon_o|\tilde{\mathbf{E}}|^2 + \frac{1}{4}\mu_o|\tilde{\mathbf{H}}|^2$ within the cavity, is then given by the ratio

$$\tau_{mnl} = \frac{W}{P_{loss}}.$$

Recall from ECE 329, Lecture 26:

Power loss per unit area of a conductor with an equivalent surface current $\tilde{\mathbf{J}}_s$ is

$$S_{loss} = \frac{1}{2} R_s |\tilde{\mathbf{J}}_s|^2$$

where

$$R_s = \sqrt{\frac{\omega\mu}{2\sigma}}$$

is the surface resistor of the conductor in terms of conductivity σ, permeability μ, and frequency ω.

3. Given the energy dissipation rate τ_{mnl} and the resonant frequency ω_{mnl}, the product

$$Q \equiv \omega_{mnl}\tau_{mnl}$$

is known as **quality factor**. Highly damped modes have small Q, while a high-Q is an indicator of a low-loss cavity.

Review the concept of Q from your ECE 210 notes (see Chpt 12).

4. In cavities with lossy walls in **thermal equilibrium** (i.e., at a steady temperature) it is observed that the average stored energy does not change with time despite the losses in the walls. What that means is that the lossy walls must be radiating as much as they absorb on the average.

The phenomenon of cavity radiation from lossy walls in thermal equilibrium — related to blackbody radiation as well as thermal resistor noise — will explored in the next lecture.

Example 1: Determine f_{nml} and $Q = \tau_{mnl}\omega_{mnl}$ for an air-filled rectangular cavity with $a = b = d = 2$ cm in TE_{101} mode.

Solution: For TE_{m0l} modes $k_y = 0$ and the field expressions derived earlier simplify as

$$\begin{aligned}
H_{zt} &= -j2\cos(k_x x)\cos(k_y y)\sin(k_z z) \to -j2\cos(k_x x)\sin(k_z z)\\
H_{xt} &= \frac{j2k_z k_x \sin(k_x x)\cos(k_y y)\cos(k_z z)}{k^2 - k_z^2} \to \frac{j2k_z \sin(k_x x)\cos(k_z z)}{k_x}\\
H_{yt} &= \frac{j2k_z k_y \cos(k_x x)\sin(k_y y)\cos(k_z z)}{k^2 - k_z^2} \to 0
\end{aligned}$$

$$E_{yt} = \frac{-2\omega\mu_o k_x \sin(k_x x)\cos(k_y y)\sin(k_z z)}{k^2 - k_z^2} \rightarrow \frac{-2\omega\mu_o \sin(k_x x)\sin(k_z z)}{k_x}$$

$$E_{xt} = \frac{2\omega\mu_o k_y \cos(k_x x)\sin(k_y y)\sin(k_z z)}{k^2 - k_z^2} \rightarrow 0.$$

Therefore, we have

$$\langle\frac{1}{2}\epsilon_o \mathbf{E}\cdot\mathbf{E}\rangle = \epsilon_o \frac{|\tilde{\mathbf{E}}|^2}{4} = \frac{\epsilon_o \omega^2 \mu_o^2 \sin^2(k_x x)\sin^2(k_z z)}{k_x^2}$$

and

$$\langle\frac{1}{2}\mu_o \mathbf{H}\cdot\mathbf{H}\rangle = \mu_o \frac{|\tilde{\mathbf{H}}|^2}{4} = \mu_o[\cos^2(k_x x)\sin^2(k_z z) + \frac{k_z^2 \sin^2(k_x x)\cos^2(k_z z)}{k_x^2}].$$

Volume integrals of these in a cavity with $a = b = c$ replace each trigonometric product in above expressions with $a^3/4$ and thus we obtain

$$W = \frac{a^3}{4}[\frac{\epsilon_o \omega^2 \mu_o^2}{k_x^2} + \mu_o[1 + \frac{k_z^2}{k_x^2}]] = \frac{a^3}{2}\frac{k^2 \mu_o}{k_x^2}$$

after using $k^2 = \omega^2 \mu_o \epsilon_o$. Also with $a = b = c$ we have $k_x^2 = k_z^2 = k^2/2$ for TE_{101} mode, and hence

$$W = a^3 \mu_o.$$

For surface currents on cavity walls we have, on top and bottom walls ($x = 0$ and $x = a$),

$$|\tilde{\mathbf{J}}_s|^2 = |\tilde{H}_z|^2 + |\tilde{H}_y|^2 = 4\sin^2(k_z z) + 0$$

on left and right walls ($y = 0$ and $y = b = a$),

$$|\tilde{\mathbf{J}}_s|^2 = |\tilde{H}_z|^2 + |\tilde{H}_x|^2 = 4\cos^2(k_x x)\sin^2(k_z z) + \frac{4k_z^2 \sin^2(k_x x)\cos^2(k_z z)}{k_x^2}$$

and on front and back walls ($z = 0$ and $z = d = a$),

$$|\tilde{\mathbf{J}}_s|^2 = |\tilde{H}_x|^2 + |\tilde{H}_y|^2 = \frac{4k_z^2 \sin^2(k_x x)}{k_x^2} + 0.$$

Integrating these three expressions over their surfaces, multiplying by 2 (two walls per each expression), and finally scaling by $R_s/2$, we obtain power loss in the walls as

$$P_{loss} = R_s a^2[2 + 1 + \frac{k_z^2}{k_x^2} + \frac{2k_z^2}{k_x^2}] = R_s a^2[2 + 1 + 1 + 2] = 6R_s a^2.$$

Finally

$$\tau = \frac{W}{P_{loss}} = \frac{a^3\mu_o}{6R_s a^2} = \frac{a\mu_o}{6R_s} = \frac{a\mu_o}{6\sqrt{\frac{\omega\mu_o}{2\sigma}}} = \frac{a}{6}\sqrt{\frac{2\sigma\mu_o}{\omega}} \;\Rightarrow\; \omega\tau = \frac{a}{6}\sqrt{2\omega\sigma\mu_o}$$

where ω is the resonance frequency for TE_{101} mode satisfying

$$k^2 = \frac{\omega^2}{c^2} = k_x^2 + k_y^2 + k_z^2 = 2(\frac{\pi}{a})^2 \;\Rightarrow\; \omega = \sqrt{2}\frac{c\pi}{a} \equiv \omega_{101}.$$

Substituting for ω above, we find that

$$\omega\tau = \frac{a}{6}\sqrt{2\omega\sigma\mu_o} = \frac{a}{6}\sqrt{2\sqrt{2}\frac{c\pi}{a}\sigma\mu_o} = \frac{1}{6}\sqrt{2\sqrt{2}c\pi a\sigma\mu_o}$$

which yields for a cavity with copper walls ($\sigma = 6 \times 10^7$ S/m)

$$\begin{aligned} Q = \omega\tau &= \frac{\sqrt{2\sqrt{2}}}{6}\sqrt{3 \times 10^8 \times \pi \times 2 \times 10^{-2} \times 6 \times 10^7 \times 4\pi \times 10^{-7}} \\ &= \frac{\sqrt{2\sqrt{2}}}{6}\sqrt{144\pi^2 \times 10^6} = \frac{\sqrt{2\sqrt{2}}}{6}12\pi \times 10^3 \approx 10.56 \times 10^3 \sim 10^4. \end{aligned}$$

Also, the resonant frequency of the mode is

$$f_{101} = \frac{\omega_{101}}{2\pi} = \frac{\sqrt{2}\frac{c\pi}{a}}{2\pi} = \frac{c}{\sqrt{2}a} = \frac{3 \times 10^8}{\sqrt{2}2 \times 10^{-2}} = \frac{30}{2\sqrt{2}}\,\mathrm{GHz} \approx 10\,\mathrm{GHz}.$$

- The result $Q = \tau\omega \sim 10^4$ from Example 1 indicates that the mode oscillates through $10^4/2\pi > 10^3$ cycles over a time period in which the mode energy decays by one e-fold.