30 Guide impedance and TL analogies

TE mode fields:

$$H_x = \frac{-jk_z \frac{\partial H_z}{\partial x}}{k^2 - k_z^2},$$
$$H_y = \frac{-jk_z \frac{\partial H_z}{\partial y}}{k^2 - k_z^2},$$
$$E_y = \frac{j\omega\mu_o \frac{\partial H_z}{\partial x}}{k^2 - k_z^2},$$
$$E_x = \frac{-j\omega\mu_o \frac{\partial H_z}{\partial y}}{k^2 - k_z^2}.$$

TM mode fields:

$$H_x = \frac{j\omega\epsilon_o \frac{\partial E_z}{\partial y}}{k^2 - k_z^2},$$
$$H_y = \frac{-j\omega\epsilon_o \frac{\partial E_z}{\partial x}}{k^2 - k_z^2},$$
$$E_y = \frac{-jk_z \frac{\partial E_z}{\partial y}}{k^2 - k_z^2},$$
$$E_x = \frac{-jk_z \frac{\partial E_z}{\partial x}}{k^2 - k_z^2}.$$

The above relations between the transverse components of TE and TM mode fields imply that

TE case:

$$\frac{E_x}{H_y} = \frac{E_y}{-H_x} = \frac{\omega\mu_o}{k_z}$$
$$= \frac{\omega\mu_o/k}{\sqrt{1 - \frac{f_c^2}{f^2}}} = \frac{\eta_o}{\sqrt{1 - \frac{f_c^2}{f^2}}} \equiv \eta_{TE}.$$

TM case:

$$\frac{E_x}{H_y} = \frac{E_y}{-H_x} = \frac{k_z}{\omega\epsilon_o}$$
$$= \frac{k\sqrt{1 - \frac{f_c^2}{f^2}}}{\omega\epsilon_o} = \eta_o\sqrt{1 - \frac{f_c^2}{f^2}} \equiv \eta_{TM}.$$

The guide impedances defined above can be used to set up transmission line models for waveguide circuits in which the parameters η_{TE} and η_{TM} for each mode play the same role as the characteristic impedance Z_o in TL theory.

- For example, two waveguides in cascade with different values of η_{TE} can be quarter-wave matched by inserting a quarter-wave section having a guide impedance equal to the geometric means of the two guides.

- For dielectric-field guides replace η_o by the appropriate η, and also in calculating the length of the quarter-wave section use $\lambda_g = \frac{2\pi}{k_z}$ appropriate for that section (see HW).

Note that, using the cutoff wavelength, we have

TE case:

$$\eta_{TE} = \frac{\eta_o}{\sqrt{1 - \frac{f_c^2}{f^2}}} = \frac{\eta_o}{\sqrt{1 - \frac{\lambda^2}{\lambda_c^2}}}$$

TM case:

$$\eta_{TM} = \eta_o \sqrt{1 - \frac{f_c^2}{f^2}} = \eta_o \sqrt{1 - \frac{\lambda^2}{\lambda_c^2}}.$$

Example 2: Consider an air-filled rectangular waveguide with $a = 3$ cm and $b = 1$ cm. Determine the TE_{10} mode fields for the guide from the results of Example 1 of Lect 29 assuming that at the operation frequency the free-space wavelength is $\lambda = 3$ cm.

Solution: By setting $k_y = 0$, $k_x = \frac{m\pi}{a} = \frac{2\pi}{\lambda_c}$, and $k_z = k\sqrt{1-(\frac{\lambda}{\lambda_c})^2} = \frac{2\pi}{\lambda_z}$ in the results of Example 1 (in Lect 29) we find for TE_{m0} mode

$$\begin{aligned}\tilde{\mathbf{H}}(x,y,z) &= H_o[\hat{x}\frac{jk_z}{k_x}\sin(k_x x) + \hat{z}\cos(k_x x)]e^{-jk_z z} \\ &= H_o[\hat{x}\frac{j\lambda_c\sqrt{1-(\frac{\lambda}{\lambda_c})^2}}{\lambda}\sin(\frac{2\pi}{\lambda_c}x) + \hat{z}\cos(\frac{2\pi}{\lambda_c}x)]e^{-jk\sqrt{1-(\frac{\lambda}{\lambda_c})^2}z}\end{aligned}$$

and

$$\begin{aligned}\tilde{\mathbf{E}}(x,y,z) &= -H_o\hat{y}\frac{j\omega\mu_o}{k_x}\sin(k_x x)e^{-jk_z z} \\ &= -H_o\hat{y}\eta_{TE}\frac{j\lambda_c\sqrt{1-(\frac{\lambda}{\lambda_c})^2}}{\lambda}\sin(\frac{2\pi}{\lambda_c}x)e^{-jk\sqrt{1-(\frac{\lambda}{\lambda_c})^2}z} \\ &= -H_o\hat{y}\eta_o\frac{j\lambda_c}{\lambda}\sin(\frac{2\pi}{\lambda_c}x)e^{-jk\sqrt{1-(\frac{\lambda}{\lambda_c})^2}z}.\end{aligned}$$

With $a = 3$ cm and $b = 1$ cm, the cutoff wavelength for TE_{10} mode is

$$\lambda_c = \frac{2a}{m} = 6\,\mathrm{cm}.$$

Thus, with $\lambda = 3$ cm

$$\sqrt{1-(\frac{\lambda}{\lambda_c})^2} = \sqrt{1-\frac{1}{4}} = \frac{\sqrt{3}}{2}\ ,\ \frac{\lambda_c}{\lambda} = 2,\ \ \frac{\lambda_z}{\lambda} = \frac{2}{\sqrt{3}}.$$

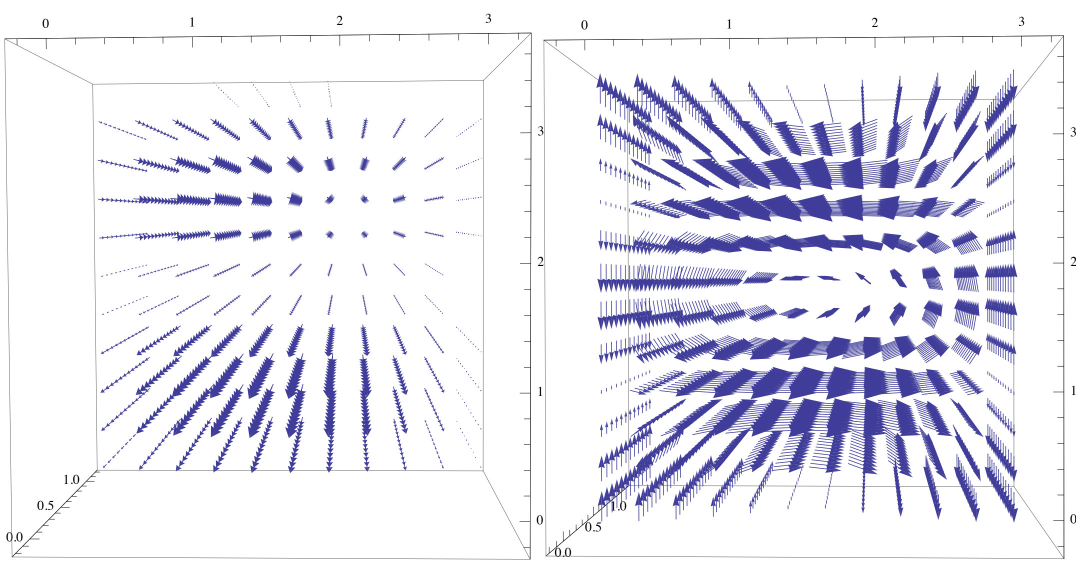
Then, for TE_{10} mode we have

$$\tilde{\mathbf{H}}(x,y,z) = H_o[\hat{x}j\sqrt{3}\sin(\frac{\pi}{3}x) + \hat{z}\cos(\frac{\pi}{3}x)]e^{-j\pi z/\sqrt{3}}$$

and

$$\tilde{\mathbf{E}}(x,y,z) = -H_o\hat{y}\eta_o j2\sin(\frac{\pi}{3}x)e^{-j\pi z/\sqrt{3}}.$$

The real part of these phasors would yield the field vectors inside the waveguide at time $t=0$, as depicted below.



- In the 3D plots shown above we depict $\mathbf{E}(x,y,z,0)$ vectors from Example 2 on the left, and $\mathbf{H}(x,y,z,0)$ on the right; the horizontal axis is x, vertical is z, and y axis is into the page (all labelled in cm units) —note that
  - there is no field variation in y-direction because this is the TE_{10} mode,
  - $\mathbf{E}\times\mathbf{H}$ is predominantly in $\hat{z}$ direction.

Example 3: Repeat Example 2 for the case of TE_{20} mode and $\lambda = 2$ cm.

Solution: For the TE_{m0} mode we have

$$\tilde{\mathbf{H}}(x,y,z) = H_o[\hat{x}\frac{j\lambda_c\sqrt{1-(\frac{\lambda}{\lambda_c})^2}}{\lambda}\sin(\frac{2\pi}{\lambda_c}x) + \hat{z}\cos(\frac{2\pi}{\lambda_c}x)]e^{-jk\sqrt{1-(\frac{\lambda}{\lambda_c})^2}z}$$

and

$$\tilde{\mathbf{E}}(x,y,z) = -H_o\hat{y}\eta_o\frac{j\lambda_c}{\lambda}\sin(\frac{2\pi}{\lambda_c}x)e^{-jk\sqrt{1-(\frac{\lambda}{\lambda_c})^2}z}.$$

With $a = 3$ cm and $b = 1$ cm, the cutoff wavelength for TE_{20} mode is

$$\lambda_c = \frac{2a}{m} = 3\,\mathrm{cm}.$$

Thus, with $\lambda = 2$ cm

$$\sqrt{1-(\frac{\lambda}{\lambda_c})^2} = \sqrt{1-\frac{4}{9}} = \frac{\sqrt{5}}{3}\ ,\frac{\lambda_c}{\lambda} = 1.5,\ \ \frac{\lambda_z}{\lambda} = \frac{3}{\sqrt{5}}.$$

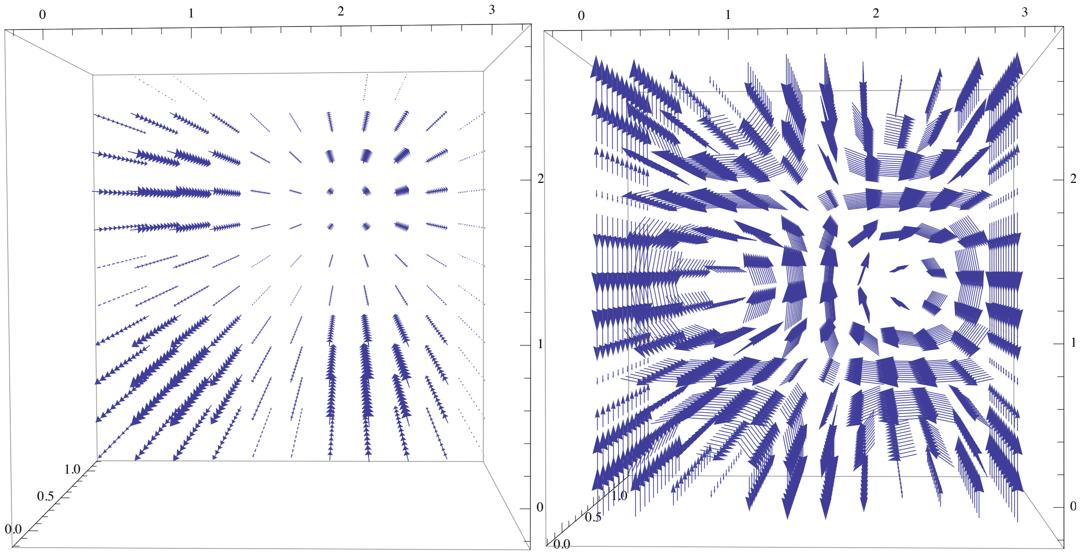
Then, for TE_{10} mode we have

$$\tilde{\mathbf{H}}(x,y,z) = H_o[\hat{x}j\frac{\sqrt{5}}{2}\sin(\frac{2\pi}{3}x) + \hat{z}\cos(\frac{2\pi}{3}x)]e^{-j\pi z\sqrt{5}/3}$$

and

$$\tilde{\mathbf{E}}(x,y,z) = -H_o\hat{y}\eta_o j\frac{3}{2}\sin(\frac{2\pi}{3}x)e^{-j\pi z\sqrt{5}/3}.$$

The real part of these phasors would yield the field vectors inside the waveguide at time $t = 0$, as depicted below.



- In the 3D plots shown above we depict $\mathbf{E}(x, y, z, 0)$ vectors from Example 3 on the left, and $\mathbf{H}(x, y, z, 0)$ on the right; the horizontal axis is x, vertical is z, and y axis is into the page (all labelled in cm units).

- Imagine the vector patterns depicted above sliding upwards in the z-axis direction at the speed $v_{pz} = \frac{\omega}{k_z}$, with each feature of the pattern passing by a stationary observer who experiences a monochromatic oscillation.

  – that would be the proper way of visualizing the propagation of an unmodulated TE_{20} mode.

Example 4: For the TE_{m0} mode we have the wave fields

$$\tilde{\mathbf{H}}(x,y,z) = H_o[\hat{x}\frac{j\lambda_c\sqrt{1-(\frac{\lambda}{\lambda_c})^2}}{\lambda}\sin(\frac{2\pi}{\lambda_c}x) + \hat{z}\cos(\frac{2\pi}{\lambda_c}x)]e^{-jk\sqrt{1-(\frac{\lambda}{\lambda_c})^2}z}$$

and

$$\tilde{\mathbf{E}}(x,y,z) = -H_o\hat{y}\eta_o\frac{j\lambda_c}{\lambda}\sin(\frac{2\pi}{\lambda_c}x)e^{-jk\sqrt{1-(\frac{\lambda}{\lambda_c})^2}z}.$$

Express the time-averaged power transmitted by the mode in $\hat{z}$ direction in terms of

$$E_o \equiv H_o\eta_o\frac{\lambda_c}{\lambda}$$

representing the amplitude of the electric field wave.

Solution: We start with the time-averaged Poynting vector

$$\begin{aligned}\langle\mathbf{E}\times\mathbf{H}\rangle &= \frac{1}{2}\mathrm{Re}\{\tilde{\mathbf{E}}\times\tilde{\mathbf{H}}^*\}\\ &= \frac{|H_o|^2\eta_o}{2}(\frac{\lambda_c}{\lambda})^2\sqrt{1-(\frac{\lambda}{\lambda_c})^2}\sin^2(\frac{2\pi}{\lambda_c}x)\hat{z}\\ &= \frac{|E_o|^2}{2\eta_o}\sqrt{1-(\frac{\lambda}{\lambda_c})^2}\sin^2(\frac{2\pi}{\lambda_c}x)\hat{z} = \frac{|E_o|^2}{2\eta_{TE}}\sin^2(\frac{2\pi}{\lambda_c}x)\hat{z}.\end{aligned}$$

Now, integrating $\langle\mathbf{E}\times\mathbf{H}\rangle\cdot\hat{z}$ across the guide cross section we get the time-average power

$$\begin{aligned}P &= \int_{x=0}^{a}\int_{y=0}^{b}\langle\mathbf{E}\times\mathbf{H}\rangle\cdot\hat{z}\,dx\,dy\\ &= \frac{|E_o|^2}{2\eta_{TE}}b\int_0^a\sin^2(\frac{2\pi}{\lambda_c}x)dx = \frac{|E_o|^2}{2\eta_{TE}}\frac{ab}{2}\end{aligned}$$

since the integral of

$$\sin^2(\frac{2\pi}{\lambda_c}x) = \frac{1}{2}(1 - \cos(\frac{4\pi}{2a/m}x)) = \frac{1}{2}(1 - \cos(2\pi m x/a))$$

yields 1/2. It can be shown that in the case of TE_{mn} modes with non-zero n, the above result for P is still valid provided $ab/2$ is replaced by $ab/4$ (see HW).

Example 5: A rectangular waveguide with $a = 2$ cm and $b = 1$ cm is air filled for $z < 0$, but is is filled with a dielectric in $z > 0$ region with a refractive index $n = 1.5$ and $\mu_r = 1$. For $f = 12.5$ GHz and TE_{10} mode operation design a $\lambda/4$ transformer to match the two sections of the waveguide. Use transmission-line analogy to solve this problem (as in Lecture 24).

Solution: To solve this problem using a transmission-line analogy we first need the impedances η_{TE} for the two sections of the guide. Since

$$\eta_{TE} = \frac{\eta}{\sqrt{1 - \frac{f_c^2}{f^2}}}$$

we need to find f_c and η in the two sections of the guide.
The cutoff frequency is

$$f_c = \frac{mc}{2a} = \frac{3 \times 10^{10}\,\text{cm/s}}{2 \times 2\,\text{cm}} = 7.5\,\text{GHz} \text{ in air,}$$

and

$$f_c = \frac{mc/n}{2a} = \frac{7.5\,\text{GHz}}{n} = \frac{7.5\,\text{GHz}}{1.5} = 5\,\text{GHz} \text{ in dielectric.}$$

Hence

$$\eta_{TE} = \frac{\eta}{\sqrt{1 - \frac{f_c^2}{f^2}}} = \frac{120\pi}{\sqrt{1 - (\frac{7.5}{12.5})^2}} = 150\pi\,\Omega \text{ in air,}$$

and

$$\eta_{TE} = \frac{\eta}{\sqrt{1 - \frac{f_c^2}{f^2}}} = \frac{120\pi/1.5}{\sqrt{1 - (\frac{5}{12.5})^2}} = \frac{400\pi}{\sqrt{21}}\,\Omega \text{ in dielectric.}$$

Since $\eta_{TE,air} \neq \eta_{TE,diel}$, we will certainly have reflections at the interface at $z = 0$ unless a matching section is inserted.

Consider a $\lambda/4$ long section of a waveguide with identical dimensions as above but filled with some dielectric having a refractive index n_x. Then, transmission-line analogy would indicate that an impedance match can be achieved if

$$\eta_{TE,air}\eta_{TE,diel} = \eta_{TE,x}^2$$

where $\eta_{TE,x}$is the impedance of the matching segment. In view of the above relations, this can be written as

$$(150\pi)(\frac{400\pi}{\sqrt{21}}) = \left[\frac{120\pi/n_x}{\sqrt{1-\left(\frac{7.5/n_x}{12.5}\right)^2}}\right]^2,$$

which yields

$$n_x^2 - \left(\frac{7.5}{12.5}\right)^2 = \frac{120^2\sqrt{21}}{150\times 400} \quad\Rightarrow\quad n_x^2 = 1.459.$$

To determine the actual length of the $\lambda/4$ long section we need to find out λ, which is really the guide wavelength λ_g for the TE_{10} mode, i.e.,

$$\begin{aligned}\lambda_g &= \frac{2\pi}{k_z} = \frac{2\pi/k}{\sqrt{1-\frac{f_c^2}{f^2}}} = \frac{c/n_x}{12.5\times 10^9\sqrt{1-\left(\frac{7.5/n_x}{12.5}\right)^2}} = \frac{30}{12.5n_x\sqrt{1-\left(\frac{7.5/n_x}{12.5}\right)^2}}\\ &= \frac{30}{\sqrt{(12.5n_x)^2-7.5^2}} = 2.28\,\mathrm{cm}.\end{aligned}$$

Thus, the matching section has a physical length of

$$d = \frac{\lambda_g}{4} = 0.572\,\mathrm{cm}.$$