29 TE$_{mn}$ modes in rectangular waveguides

- The analysis of TE$_{mn}$ modes starts with the wave equation for H_z, that is

$$\nabla^2 H_z + k^2 H_z = 0.$$

By analogy to the TM$_{mn}$ case, and using separation of variables, we have

$$H_z(x,y,z) = (A\cos k_x x + B\sin k_x x)(C\cos k_y y + D\sin k_y y)e^{-jk_z z}.$$

Pertinent boundary conditions need to be applied in terms of E_y and E_x on waveguide walls at $x=0$ and a, and $y=0$ and b, respectively:

1. $E_y = 0$ at $x=0$ and a requires $\frac{\partial H_z}{\partial x} = 0$ at the same locations, implying $B=0$ and $k_x a = m\pi$.
2. $E_x = 0$ at $y=0$ and b requires $\frac{\partial H_z}{\partial y} = 0$ at the same locations, implying $D=0$ and $k_y b = n\pi$.

Hence,

$$H_z(x,y,z) = H_o\cos(k_x x)\cos(k_y y)e^{-jk_z z},$$

with

$$k_x = \frac{m\pi}{a},\quad k_y = \frac{n\pi}{b},\quad k_z = \frac{\omega}{c}\sqrt{1-\frac{k_x^2+k_y^2}{k^2}} = \frac{\omega}{c}\sqrt{1-\frac{f_c^2}{f^2}},$$

TE mode fields:

$$E_x = \frac{-j\omega\mu_o\frac{\partial H_z}{\partial y}}{k^2-k_z^2},$$

$$E_y = \frac{j\omega\mu_o\frac{\partial H_z}{\partial x}}{k^2-k_z^2},$$

$$H_x = \frac{-jk_z\frac{\partial H_z}{\partial x}}{k^2-k_z^2},$$

$$H_y = \frac{-jk_z\frac{\partial H_z}{\partial y}}{k^2-k_z^2}.$$

where

$$f_c = \sqrt{(\frac{mc}{2a})^2 + (\frac{nc}{2b})^2}$$

is the pertinent cutoff frequency of the TE_{mn} mode.

- Note that $m = 0$ or $n = 0$ — but not both zero — are permitted since these choices do not lead to trivial H_z.
- However, $m = n = 0$ is not permitted, because in that case H_z becomes independent of x and y, and leads to zero transverse fields.

Example 1: Determine the transverse field components for the TE_{mn} mode explicitly by differentiating

$$H_z(x, y, z) = H_o \cos(k_x x) \cos(k_y y) e^{-jk_z z}$$

and using the relations in the margin. Show that the fields for TE_{00} are trivial while the fields TE_{m0} are finite.

Solution: We have,

$$H_x = \frac{-jk_z \frac{\partial H_z}{\partial x}}{k^2 - k_z^2} = \frac{jk_z H_o k_x \sin(k_x x) \cos(k_y y) e^{-jk_z z}}{k_x^2 + k_y^2},$$

$$H_y = \frac{-jk_z \frac{\partial H_z}{\partial y}}{k^2 - k_z^2} = \frac{jk_z H_o k_y \cos(k_x x) \sin(k_y y) e^{-jk_z z}}{k_x^2 + k_y^2},$$

TE mode fields:

$$E_x = \frac{-j\omega\mu_o \frac{\partial H_z}{\partial y}}{k^2 - k_z^2},$$

$$E_y = \frac{j\omega\mu_o \frac{\partial H_z}{\partial x}}{k^2 - k_z^2},$$

$$H_x = \frac{-jk_z \frac{\partial H_z}{\partial x}}{k^2 - k_z^2},$$

$$H_y = \frac{-jk_z \frac{\partial H_z}{\partial y}}{k^2 - k_z^2}.$$

$$E_y = \frac{j\omega\mu_o \frac{\partial H_z}{\partial x}}{k^2 - k_z^2} = \frac{-j\omega\mu_o H_o k_x \sin(k_x x)\cos(k_y y)e^{-jk_z z}}{k_x^2 + k_y^2},$$

$$E_x = \frac{-j\omega\mu_o \frac{\partial H_z}{\partial y}}{k^2 - k_z^2} = \frac{j\omega\mu_o H_o k_y \cos(k_x x)\sin(k_y y)e^{-jk_z z}}{k_x^2 + k_y^2}.$$

For TE_{m0} we have $k_y = \frac{n\pi}{b} = 0$ and, therefore,

$$H_x = \frac{jk_z H_o \sin(k_x x)e^{-jk_z z}}{k_x}, \quad H_y = 0, \quad E_y = \frac{-j\omega\mu_o H_o \sin(k_x x)e^{-jk_z z}}{k_x}, \quad E_x = 0.$$

Now, we obtain the TE_{00} field from these by setting $k_x = 0$ using L'Hospital's law, leading to

$$H_x = jk_z H_o x e^{-jk_z z}, \quad H_y = 0, \quad E_y = -j\omega\mu_o H_o x e^{-jk_z z}, \quad E_x = 0;$$

these violate the boundary condition of zero E_y at $x = a$ unless $H_o = 0$, which is of course the trivial solution.

Waveguide design and application examples:

Example 2: Design a rectangular air-filled wave guide for single-mode transmission of the frequency band 3.75 GHz – 4.25 GHz in TE_{10} mode. That is, select the dimensions a and $b \leq a$ of the waveguide so that only the TE_{10} mode is propagating in the guide within the specified frequency band while cross sectional area ab is as large as possible for purposes of the power transmission capacity of the guide.

Solution: First, to make sure that TE_{10} mode is propagating in the band for $f > 3.75$ GHz, we need

$$f_c = \sqrt{(\frac{mc}{2a})^2 + (\frac{nc}{2b})^2}\Big|_{\substack{m=1\\ n=0}} = \frac{c}{2a} < 3.75 \times 10^9\,\mathrm{Hz}$$

from which we get

$$a > \frac{3 \times 10^{10}\,\mathrm{cm/s}}{2 \times 3.75 \times 10^9\,/\mathrm{s}} = 4\,\mathrm{cm}.$$

With $a = 4$ cm, the cutoff frequency of TE_{20} mode will be 7.5 GHz, which is safely outside our band of interest. Of course with $a > 4$ cm TE_{20} cutoff frequency will be less than 7.5 GHz, and we can afford reducing it to as small as 4.25 GHz by selecting

$$a = \frac{cm}{2f_{cm0}}\Big|_{m=2} = \frac{3 \times 10^{10} \times 2}{2 \times 4.25 \times 10^9} = \frac{30}{4.25} = 7.06\,\mathrm{cm}.$$

To ensure single mode operation in 3.75 GHz – 4.25 GHz band we also need for TE_{01} mode a cutoff frequency

$$f_c = \sqrt{(\frac{mc}{2a})^2 + (\frac{nc}{2b})^2}\Big|_{\substack{m=0\\ n=1}} = \frac{c}{2b} > 4.25 \times 10^9\,\mathrm{Hz}$$

yielding

$$b < \frac{3 \times 10^{10}\,\mathrm{cm/s}}{2 \times 4.25 \times 10^{9}\,/\mathrm{s}} = 3.53\,\mathrm{cm}.$$

Hence, a design with maximum possible ab for the specified band works out to have $a = 7.06$ cm and $b = a/2 = 3.53$ cm.

Example 3: Re-design the waveguide in Example 2 for the frequency band 3.75 GHz – 4.25 GHz to include some safety margins as follows: Select the dimensions of the wave guide such that the lowest frequency of the band is at least 20% above the cutoff frequency of the fundamental mode (TE_{10}), and the highest frequency of the band is at least 20% lower than the cutoff frequency of the next higher-order mode.

Solution: We already have the lowest frequency of the band, 3.75 GHz, more than 20% above the TE_{10} cutoff frequency $c/2a = 2.125$ GHz — therefore at first it appears that $a = 7.06$ cm can remain as is.

But b clearly has to change. To select b, let 4.25 GHz be 0.8 times the cutoff frequency of the TE_{01} mode. Hence

$$4.25 \times 10^{9} = 0.8\frac{c}{2b} \Rightarrow \quad b = \frac{0.8 \times 3 \times 10^{10}\,\mathrm{cm/s}}{2 \times 4.25 \times 10^{9}\,/\mathrm{s}} = 2.82\,\mathrm{cm}.$$

But then we realize that with $a = 7.06$ cm, 4.25 GHz is still the cutoff frequency the of the TE_{20} mode, which is no longer permissible because a safety margin is needed — TE_{20} cutoff frequency also needs to be moved up by the same margin as TE_{01}; this can be realized by taking the new modified a as $a = 2b = 5.64$ cm. The corresponding TE_{10} mode cutoff frequency is

$$f_c = \frac{mc}{2a}\Big|_{m=1} = \frac{c}{2a} = \frac{c}{4b} = \frac{30 \times 10^9}{4 \times 2.82} = 2.65\,\text{GHz}$$

and 3.75 GHz is still more than 20% above this.

In conclusion, with $a = 5.64$ cm and $b = 2.82$ cm we have the required modified dimensions and safety margins.

Example 4: The waveguide of Example 3 is to be used as an attenuator for the next (non-propagating) higher-order mode. What is the minimum attenuation rate for the mode in dB/cm over the band 3.75 GHz – 4.25 GHz?

Solution: The next higher-order modes are TE_{01} and TE_{20} having equal cutoff frequencies because $a = 2b$.
The attenuation of these modes will be less severe at $f = 4.25$ GHz than at 3.75 GHz. We have, for these modes, at $f = 4.25$ GHz,

$$k_z = k\sqrt{1 - (\frac{f_c}{f})^2} = k\sqrt{1 - (\frac{1}{0.8})^2} = k\sqrt{1 - (\frac{5}{4})^2} = -jk\frac{3}{4}.$$

Since, at $f = 4.25$ GHz,

$$\lambda = \frac{3 \times 10^{10}}{4.25 \times 10^9} = \frac{30}{4.25}\text{ cm} \quad \Rightarrow \quad k = \frac{2\pi}{\lambda} = \frac{4.25\pi}{15}\,\text{rad/cm},$$

we have

$$|k_z| = \frac{3}{4}k = \frac{3}{4}\frac{4.25\pi}{15} = \frac{4.25\pi}{20}\frac{\text{Np}}{\text{cm}}.$$

Consequently, the attenuation rate is

$$20\log_{10} e^{|k_z|} = 4.25\pi\log_{10} e = 5.7986\,\text{dB/cm}.$$