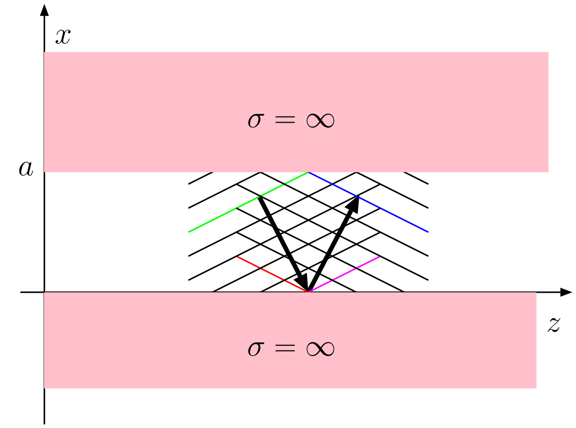
26 Parallel-plate waveguides — TM_m modes



- Last lecture we discussed the TE_m modes of propagation in parallel-plate waveguides.

- These guided modes have y-polarized electric fields transverse to the propagation direction z and exhibit a standing wave pattern in x-direction with m half-wavelengths of variation between the guide plates at $x = 0$ and $x = a$.

  – More specifically, the TE_m modes have transverse electric field phasors

$$\tilde{\mathbf{E}} = 2j\hat{y}E_o e^{-jk_z z}\sin(k_x x)$$

  where

$$k_x = \frac{m\pi}{a}, \quad m = 1, 2, \cdots$$

  and

$$k_z = \sqrt{k^2 - k_x^2} = \frac{\omega}{c}\sqrt{1 - \frac{f_c^2}{f^2}}$$

  with **cutoff frequencies**

$$f_c = \frac{mc}{2a}.$$

  Alternatively (and equivalently),

$$k_z = \sqrt{k^2 - k_x^2} = \frac{\omega}{c}\sqrt{1 - \frac{\lambda^2}{\lambda_c^2}},$$

with **cutoff wavelengths**

$$\lambda_c = \frac{2a}{m}.$$

Above, the operation frequency f and operation wavelength λsatisfy $\lambda f = c$, and furthermore

$$k = \frac{2\pi}{\lambda} = \frac{\omega}{c}$$

is the operation wavenumber. The propagation characteristics of the guided mode, on the other hand, depends on k_z, with

$$v_p = \frac{\omega}{k_z} \text{ and } \lambda_g = \frac{2\pi}{k_z}$$

denoting the phase velocity and the wavelength of the guided mode when

$$f > f_c \text{ and, equivalently, } \lambda < \lambda_c,$$

corresponding to propagation condition for a given mode. When

$$f < f_c \text{ and, equivalently, } \lambda > \lambda_c,$$

the mode is evanescent.

- Since

$$k_z = \sqrt{k^2 - k_x^2} = \frac{\omega}{c}\sqrt{1 - \frac{f_c^2}{f^2}}$$

is effectively the dispersion relation of the guided modes having the same form as the plasma dispersion relation, it follows that the group velocity is

$$v_g = \frac{\partial \omega}{\partial k_z} = c\sqrt{1 - \frac{f_c^2}{f^2}} \text{ and } v_g v_p = c^2$$

just like in plasmas.

– Finally TE_m mode fields have a guide impedance

$$\eta_{TE} = -\frac{E_y}{H_x} = \frac{\eta_o}{\sqrt{1 - \frac{f_c^2}{f^2}}}$$

relating the transverse field components of the wave.

- Next we turn our attention on TM_m mode fields which share most of the dispersion characteristics of the TE_m mode fields. However, they are essentially orthogonal to TE_m mode fields and furthermore support the $m = 0$ case which is absent for TE_m modes.

- TM_m mode guided waves propagating in z direction correspond to superpositions of incident and reflected TEM plane waves with

$$\tilde{\mathbf{H}}_i = \hat{y} H_o e^{-j(-k_x x + k_z z)}$$

and

$$\tilde{\mathbf{H}}_r = \hat{y} H_o \Gamma e^{-j(k_x x + k_z z)}$$

where $\Gamma = R = 1$ is the TM-mode reflection coefficient at an air-PEC interface.

For permissible TM_m modes $\tilde{\mathbf{H}}_r$ gets reflected (once more) *at* $x = a$ to become $\tilde{\mathbf{H}}_i$ at the same location, and thus it is necessary that

$$(\hat{y} H_o \Gamma e^{-j(k_x a + k_z z)})\Gamma = \hat{y} H_o e^{-j(-k_x a + k_z z)} e^{-j2\pi m}$$

for any integer m, i.e.,

$$|\Gamma|^2 e^{j2\angle\Gamma} e^{-jk_x a} = e^{jk_x a} e^{-j2\pi m}.$$

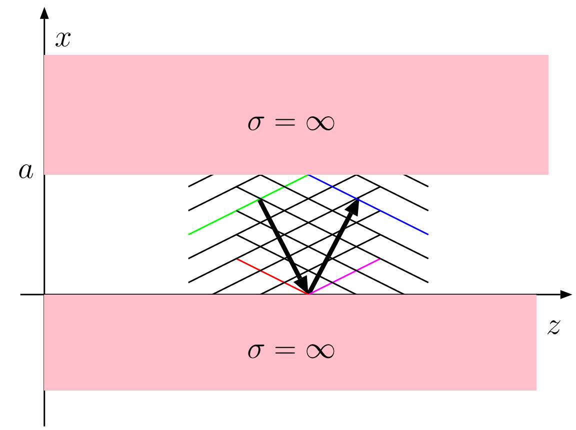
This is possible, since $|\Gamma| = |R| = 1$ and $\angle\Gamma = \angle R = 0$, with

$$-k_x a = k_x a - 2\pi m \;\;\Rightarrow\;\; k_x a = m\pi,$$

leading to

$$k_x = \frac{m\pi}{a}, \;\; m = 0, 1, 2, \cdots$$

as the guiding condition for TM_m modes.



- Since for TM_m modes the transverse field

$$\tilde{\mathbf{H}} = \tilde{\mathbf{H}}_i + \tilde{\mathbf{H}}_r = 2\hat{y}H_o e^{-jk_z z}\cos(k_x x)$$

does not vanish with vanishing k_x, the $m = 0$ mode is permitted. In fact, TM_0 mode corresponding to $m = 0$ is the TEM mode studied in EEC 329 in transmission line (TL) theory.

  - TM_0=TEM consists of wave fields

$$\tilde{\mathbf{E}} = \hat{x}E_o e^{-jkz} \quad \text{and} \quad \tilde{\mathbf{H}} = \hat{y}\frac{E_o}{\eta_o}e^{-jkz}$$

  which naturally satisfy the boundary conditions at $x = 0$ and $x = a$ planes of having zero tangential electric field.

  - Also, for this mode

$$k_x = 0 \text{ and } k_z = k,$$

  which follows when $m = 0$ is permitted in dispersion equations when applied for the case of TM_m modes.

Example 1: TM_m mode fields have transverse magnetic intensity phasors

$$\tilde{\mathbf{H}} = 2\hat{y}H_o e^{-jk_z z}\cos(k_x x).$$

(a) Determine the electric field phasor $\tilde{\mathbf{E}}$ for TM_m mode waves. (b) Also determine $\eta_{TM} \equiv \frac{E_x}{H_y}$, the effective guide impedance for TM_m mode.

Solution: (a) Using Ampere's law, we have

$$\begin{aligned}\tilde{\mathbf{E}} &= \frac{\nabla \times \tilde{\mathbf{H}}}{j\omega\epsilon_o} = \frac{\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & H_y & 0 \end{vmatrix}}{j\omega\epsilon_o} = \frac{-\hat{x}\frac{\partial H_y}{\partial z} + \hat{z}\frac{\partial H_y}{\partial x}}{j\omega\epsilon_o} \\ &= \frac{2H_o}{j\omega\epsilon_o}(\hat{x}(jk_z\cos(k_x x) - \hat{z}k_x\sin(k_x x))e^{-jk_z z}.\end{aligned}$$

(b) Using the result of part (a), we have

$$\begin{aligned}\eta_{TM} &= \frac{E_x}{H_y} = \frac{\frac{2H_o}{j\omega\epsilon_o}jk_z\cos(k_x x)e^{-jk_z z}}{2H_o e^{-jk_z z}\cos(k_x x)} \\ &= \frac{k_z}{\omega\epsilon_o} = \frac{\frac{\omega}{c}\sqrt{1-\frac{f_c^2}{f^2}}}{\omega\epsilon_o} = \eta_o\sqrt{1-\frac{f_c^2}{f^2}}.\end{aligned}$$

- Note that the results obtained in Example 1 give non-trivial results for $m = 0$ case with $k_x = 0$ and $k_z = k$.

- TM_0=TEM mode has no cutoff frequency and it is non-dispersive. It has all the properties of the unguided TEM waves we are familiar with.

- Finally, regarding dispersive TE_m and TM_m modes with $m \geq 1$, all the equations derived above can also be used when the guiding plates are embedded in dielectric media (instead of air) by simply replacing

$$c = \frac{1}{\sqrt{\mu_o \epsilon_o}} \text{ with } v_p = \frac{1}{\sqrt{\mu\epsilon}}$$

in the dispersion equations.

- There is a straightforward geometrical interpretation of v_g obtained for guided TE and TM modes.

  – Clearly the component TEM waves which constitute the guided modes (TE and TM) propagate at angles $\pm\theta$ with a velocity c in air-filled waveguides. The projection along z of the velocity vectors pointing in $\pm\theta$ directions are

$$c \sin\theta = c\sqrt{1-\cos^2\theta} = c\sqrt{1-\frac{k_x^2}{k^2}} = c\sqrt{1-\frac{f_c^2}{f^2}},$$

  which is of course the group velocity of the guided modes as we have seen before.

This makes sense: in the component TEM waves — which are non-dispersive — of the guided modes, the phase fronts as well as any imposed modulations move with the same velocity, namely c.

– While the progress of modulation on the component waves along $\pm\theta$ occurs at a velocity c, the modulation covers a shorter distance along z than the corresponding slant distance along $\pm\theta$, and thus v_g measuring the progress of the modulation along z is smaller than c measuring the same progress along $\pm\theta$.