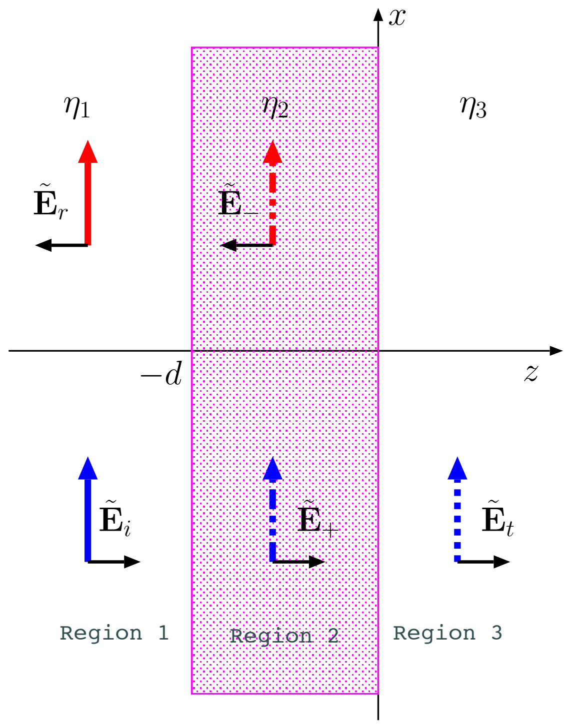
24 Evanescent waves and tunneling



- In this lecture we will explore the tunneling phenomenon associated with evanescent waves established within finite-width regions.

  The multi-slab tunneling result to be derived in this lecture will:

  1. Enhance our qualitative understanding of the frustrated-TIR example shown back in Lecture 19,
  2. Illustrate a methodology based on ***transmission line analogies*** to be used in forthcoming lectures on waveguides.

- Consider the three-slab geometry depicted in the margin where a TEM wave field

$$\tilde{\mathbf{E}}_i = \hat{x}E_i e^{-jk_1 z}, \text{ accompanied by } \tilde{\mathbf{H}}_i = \hat{y}\frac{E_i}{\eta_1}e^{-jk_1 z},$$

is incident from the left in the region $z < -d$ (region 1). As a response a reflected wave

$$\tilde{\mathbf{E}}_r = \hat{x}E_r e^{jk_1 z}, \text{ accompanied by } \tilde{\mathbf{H}}_r = -\hat{y}\frac{E_r}{\eta_1}e^{jk_1 z},$$

is set up in the same region, as well as

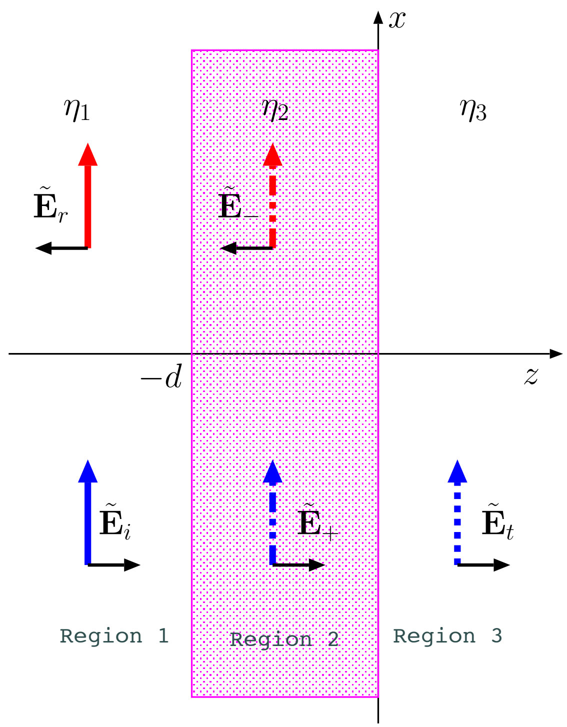
$$\tilde{\mathbf{E}}_+ = \hat{x}E_+ e^{-jk_2 z}, \text{ accompanied by } \tilde{\mathbf{H}}_+ = \hat{y}\frac{E_+}{\eta_2}e^{-jk_2 z},$$

and

$$\tilde{\mathbf{E}}_- = \hat{x}E_- e^{jk_2 z}, \text{ accompanied by } \tilde{\mathbf{H}}_- = -\hat{y}\frac{E_-}{\eta_2}e^{jk_2 z},$$

in the region $-d < z < 0$ (region 2). Finally, in region $z > 0$, we will have

$$\tilde{\mathbf{E}}_t = \hat{x} E_t e^{-jk_3 z}, \text{ accompanied by } \tilde{\mathbf{H}}_t = \hat{y} \frac{E_t}{\eta_3} e^{-jk_3 z}.$$



- Our aim is to determine the amplitudes E_t, E_+, E_-, E_r in terms of E_i using tangential boundary conditions at $z = -d$ and $z = 0$.
- We are in particular interested in the ratio of the transmitted power in region 3 to the incident power in region 1 as a function of slab width d as well as the refractive indices n_1, n_2, and n_3, including the case when n_2 is purely imaginary, the case corresponding to region 2 being in evanescent mode.

• Starting with the boundary at $z = 0$, the continuity of tangential $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{H}}$ across the boundary requires that

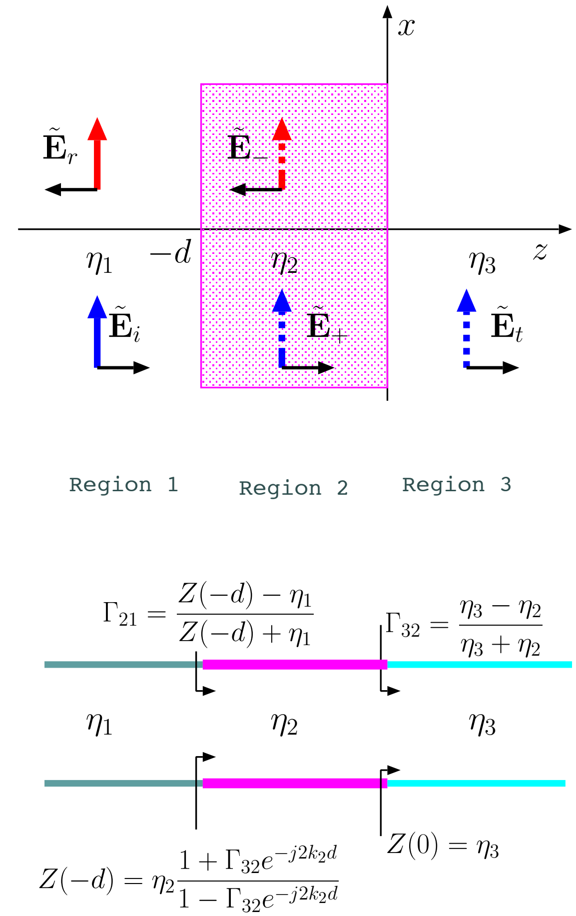
$$E_+ + E_- = E_t \text{ and } \frac{E_+ - E_-}{\eta_2} = \frac{E_t}{\eta_3}.$$

These equations can be solved for E_t and E_- in terms of E_+ to obtain

$$E_t = \underbrace{\frac{2\eta_3}{\eta_3 + \eta_2}}_{\tau_{32}} E_+ \text{ and } E_- = \underbrace{\frac{\eta_3 - \eta_2}{\eta_3 + \eta_2}}_{\Gamma_{32}} E_+.$$

Note that we have defined a pair of coefficients representing the interaction at $z = 0$ interface: a **transmission coefficient** τ_{32} and a

reflection coefficient Γ_{32} in terms of intrinsic impedances η_3 and η_2 in a manner analogous to similar relations seen in our studies of *transmission line* (TL) systems (in ECE 329).



- At the boundary on $z = -d$ plane the continuity of tangential $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{H}}$ requires that

$$E_i e^{jk_1d} + E_r e^{-jk_1d} = E_+ e^{jk_2d} + E_- e^{-jk_2d}$$

and

$$\frac{E_i e^{jk_1d} - E_r e^{-jk_1d}}{\eta_1} = \frac{E_+ e^{jk_2d} - E_- e^{-jk_2d}}{\eta_2}$$

respectively. To utilize these relations in a close analogy to TL problems we next define an effective **field impedance** $Z(-d)$ for the $z = -d$ plane as

$$Z(-d) \equiv \frac{E_+ e^{jk_2d} + E_- e^{-jk_2d}}{\frac{E_+ e^{jk_2d} - E_- e^{-jk_2d}}{\eta_2}} = \eta_2 \frac{1 + \frac{E_-}{E_+} e^{-j2k_2d}}{1 - \frac{E_-}{E_+} e^{-j2k_2d}} = \eta_2 \frac{1 + \Gamma_{32} e^{-j2k_2d}}{1 - \Gamma_{32} e^{-j2k_2d}}.$$

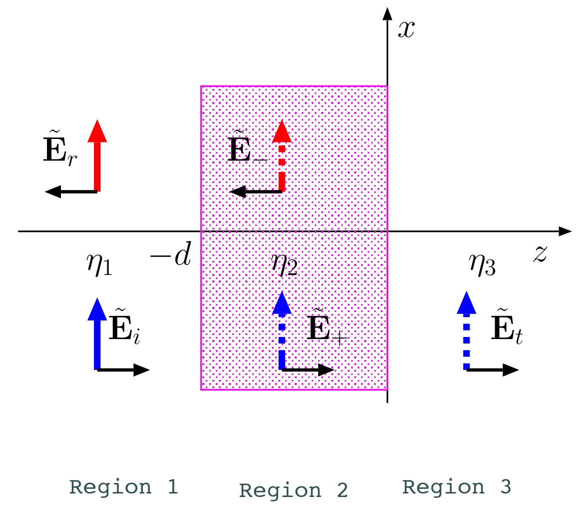
But, by the boundary condition equations above it is also true that

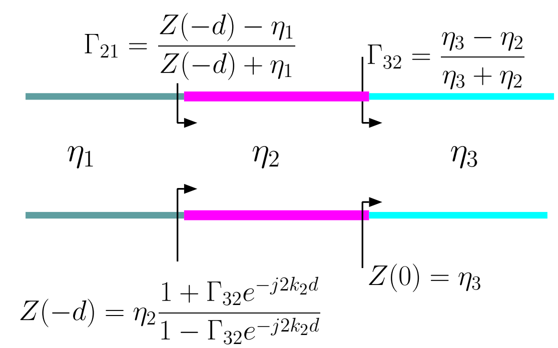
$$Z(-d) = \frac{E_i e^{jk_1d} + E_r e^{-jk_1d}}{\frac{E_i e^{jk_1d} - E_r e^{-jk_1d}}{\eta_1}} = \eta_1 \frac{1 + \Gamma_{21}}{1 - \Gamma_{21}} \quad \text{where} \quad \Gamma_{21} \equiv \frac{E_r e^{-jk_1d}}{E_i e^{jk_1d}}.$$

Solving the above expression for the **reflection coefficient** Γ_{21} at $z = -d$ plane in terms of impedance $Z(-d)$ we find that

$$\Gamma_{21} = \frac{Z(-d) - \eta_1}{Z(-d) + \eta_1}.$$

- The parameters Γ_{32}, $Z(-d)$, and Γ_{21} introduced above, bearing a strong analogy to an equivalent TL problem suggested in the margin, are sufficient to calculate the reflected and transmitted powers in our multiple slab problem as follows:





1. We first note that

$$|\Gamma_{21}|^2 = |\frac{E_r e^{-jk_1 d}}{E_i e^{jk_1 d}}|^2 \quad \Rightarrow \quad \frac{\langle S_r \rangle}{\langle S_i \rangle} = \frac{|E_r|^2/2\eta_1}{|E_i|^2/2\eta_1} = |\Gamma_{21}|^2$$

gives the **reflectance**, the fraction of the time-averaged incident power density reflected by the slab discontinuity back into region 1.

2. Assuming that the slab in region 2 is lossless, the **transmittance**, the time-averaged power density transmitted into the region 3 has to be

$$\langle S_t \rangle = \langle S_i \rangle - \langle S_r \rangle = \langle S_i \rangle (1 - |\Gamma_{21}|^2) \quad \Rightarrow \quad \frac{\langle S_t \rangle}{\langle S_i \rangle} = \frac{|E_t|^2/2\eta_3}{|E_i|^2/2\eta_1} = 1 - |\Gamma_{21}|^2.$$

The upshot is

$$\frac{\langle S_r \rangle}{\langle S_i \rangle} = |\Gamma_{21}|^2 \text{ and } \frac{\langle S_t \rangle}{\langle S_i \rangle} = 1 - |\Gamma_{21}|^2$$

where

$$\Gamma_{21} = \frac{Z(-d) - \eta_1}{Z(-d) + \eta_1}, \quad Z(-d) = \eta_2 \frac{1 + \Gamma_{32} e^{-j2k_2 d}}{1 - \Gamma_{32} e^{-j2k_2 d}}, \quad \Gamma_{32} = \frac{\eta_3 - \eta_2}{\eta_3 + \eta_2}$$

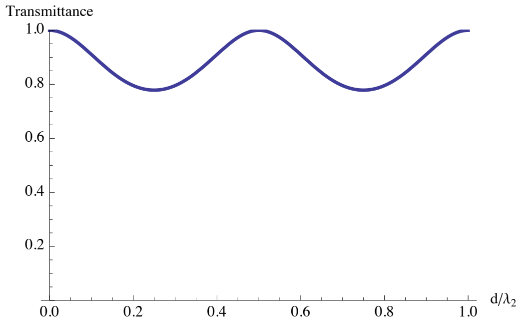
in analogy with an equivalent TL problem. An extension of these relations to an n-slab configuration is straightforward.

Example 1: Assume that regions 1 and 3 are free space whereas region 2 is a plasma slab of some width d and a plasma frequency f_p. Determine and plot the transmittance

$$\frac{\langle S_t \rangle}{\langle S_i \rangle} = 1 - |\Gamma_{21}|^2$$

as a function of d if (a) $f = \frac{5}{4} f_p$, and (b) $f = \frac{4}{5} f_p$.

Transmittance curve for part (a) when region 2 is in propagation mode:



Solution: (a) In this case the plasma refractive index in the slab is

$$n_2 = \sqrt{1 - \frac{f_p^2}{f^2}} = \sqrt{1 - (\frac{4}{5})^2} = \sqrt{1 - \frac{16}{25}} = \frac{3}{5}.$$

Hence, with $\eta_1 = \eta_3 = \eta_o$ and $\eta_2 = \eta_o / n_2 = 5\eta_o/3$, we have

$$\Gamma_{32} = \frac{\eta_o - \frac{5}{3}\eta_o}{\eta_o + \frac{5}{3}\eta_o} = \frac{3-5}{3+5} = -\frac{2}{8} = -0.25$$

also, with real $k_2 = \frac{2\pi}{\lambda_2}$, we have

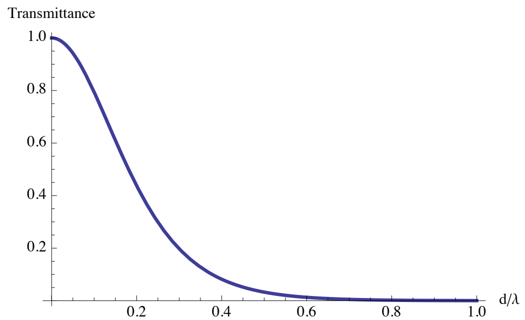
$$Z(-d) = \eta_2 \frac{1 + \Gamma_{32} e^{-j2k_2 d}}{1 - \Gamma_{32} e^{-j2k_2 d}} = \frac{5}{3}\eta_o \frac{1 - 0.25 e^{-j4\pi \frac{d}{\lambda_2}}}{1 + 0.25 e^{-j4\pi \frac{d}{\lambda_2}}};$$

thus

$$\Gamma_{21} = \frac{Z(-d) - \eta_1}{Z(-d) + \eta_1} = \frac{\frac{5}{3}\frac{1 - 0.25 e^{-j4\pi \frac{d}{\lambda_2}}}{1 + 0.25 e^{-j4\pi \frac{d}{\lambda_2}}} - 1}{\frac{5}{3}\frac{1 - 0.25 e^{-j4\pi \frac{d}{\lambda_2}}}{1 + 0.25 e^{-j4\pi \frac{d}{\lambda_2}}} + 1}.$$

A plot of the transmittance $1 - |\Gamma_{21}|^2$ versus d/λ_2 is shown in the margin. The transmittance shows a $\lambda_2/2$ periodicity in slab width d in consistency with the periodicity expected for lossless TL systems.

Transmittance curve for part (b) when region 2 is in evanescence mode:



Note that adjusting d/λ to about 0.2 sets the transmittance as 1/2, creating in effect a "beam splitter" in reference to our discussions of prisms and tunneling in Lecture 24.

Solution: (b) In this case the plasma refractive index in the slab is

$$n_2 = \sqrt{1 - \frac{f_p^2}{f^2}} = \sqrt{1 - (\frac{5}{4})^2} = \sqrt{1 - \frac{25}{16}} = \sqrt{-\frac{9}{16}} = \pm j\frac{3}{4}$$

Hence, with $\eta_1 = \eta_3 = \eta_o$ and $\eta_2 = \eta_o/n_2 = j\frac{4}{3}\eta_o$, we have

$$\Gamma_{32} = \frac{\eta_o - j\frac{4}{3}\eta_o}{\eta_o + j\frac{4}{3}\eta_o} = \frac{3 - 4j}{3 + 4j}$$

with unity magnitude, a consequence of *evanescence* in region 2; also, $k_2 = kn_2 = -j3k/4$, and we have

$$Z(-d) = \eta_2 \frac{1 + \Gamma_{32} e^{-j2k_2 d}}{1 - \Gamma_{32} e^{-j2k_2 d}} = j\frac{4}{3}\eta_o \frac{(3+4j) + (3-4j)e^{-3\pi d/\lambda}}{(3+4j) - (3-4j)e^{-3\pi d/\lambda}};$$

thus

$$\Gamma_{21} = \frac{Z(-d) - \eta_1}{Z(-d) + \eta_1} = \frac{j\frac{4}{3}\frac{(3+4j)+(3-4j)e^{-3\pi d/\lambda}}{(3+4j)-(3-4j)e^{-3\pi d/\lambda}} - 1}{j\frac{4}{3}\frac{(3+4j)+(3-4j)e^{-3\pi d/\lambda}}{(3+4j)-(3-4j)e^{-3\pi d/\lambda}} + 1}.$$

A plot of transmittance $1 - |\Gamma_{21}|^2$ versus d/λ is shown in the margin. Note the strong **tunneling effect** at small d/λ.

- A fascinating aspect of tunneling is:
  - even though the time-averaged Poynting vectors — i.e., the avg power densities — associated with the evanescent wave fields $\tilde{\mathbf{E}}_+$ and $\tilde{\mathbf{E}}_-$ in region 2 are individually *zero* because of the 90° phase shift between

$$\tilde{\mathbf{E}}_+ \text{ and } \tilde{\mathbf{H}}_+ \quad \text{as well as} \quad \tilde{\mathbf{E}}_- \text{ and } \tilde{\mathbf{H}}_-,$$

    the time-averaged Poynting vector associated with $\tilde{\mathbf{E}}_+ + \tilde{\mathbf{E}}_-$, i.e.,

$$\frac{1}{2}\mathrm{Re}\{(\tilde{\mathbf{E}}_+ + \tilde{\mathbf{E}}_-) \times (\tilde{\mathbf{H}}_+ + \tilde{\mathbf{H}}_-)^*\}$$

    pertinent for region 2, is (as shown in HW) *non-zero* (and independent of position within region 2) because of the *non-zero cross term contributions* between

$$\tilde{\mathbf{E}}_+ \text{ and } \tilde{\mathbf{H}}_- \quad \text{as well as} \quad \tilde{\mathbf{E}}_- \text{ and } \tilde{\mathbf{H}}_+.$$

    - By contrast, in propagating regions (i.e., non-evanescent), the cross product terms cancel while "self product" terms determine the net average Poynting vector.

There are many practical implications and applications of tunneling:

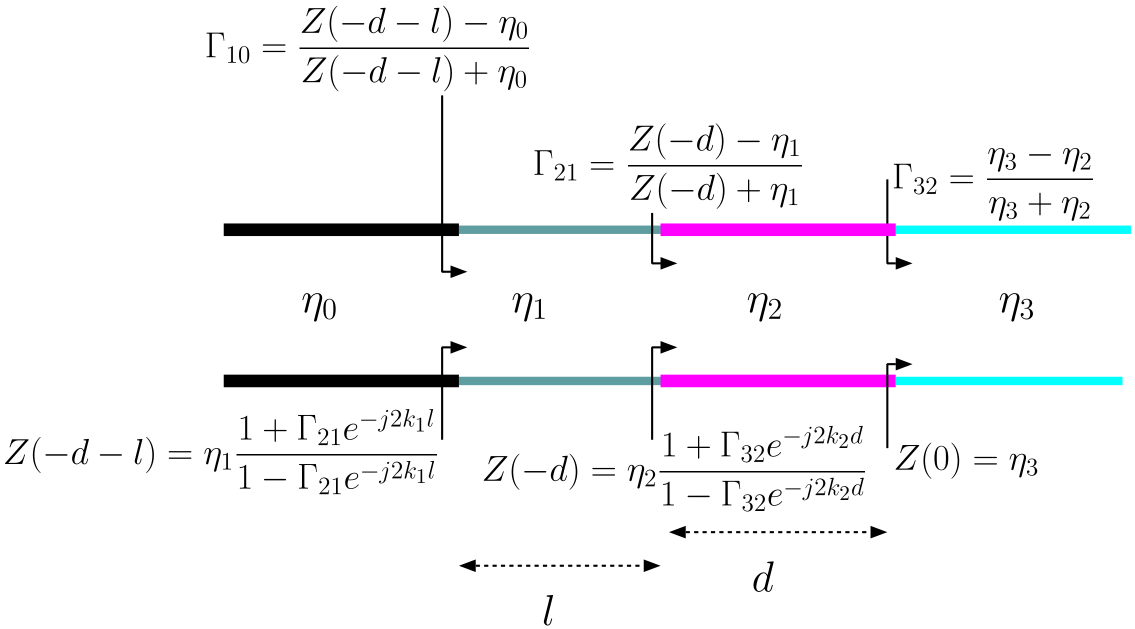
  - Beam splitters, attenuators, (undesired) interference effect due to coupling of nearby systems ...

Quantum mechanical tunneling:

In quantum physics one talks about *probabilities* of encountering particles in a given physical system rather than the particle trajectories; furthermore, the probabilities are calculated as magnitude squares (like the average power) of "wave functions" obeying a wave equation (e.g., *Schroedinger's equation* in case of non-relativistic particles). Since waves in general (including Schrodinger waves) can exhibit tunneling properties across evanescent regions (as shown in this section), finite probabilities can be calculated in quantum mechanics for particles in regions separated from their source regions by classically impenetrable barriers (in which the wave function is evanescent).

Phenomena such as radioactive decay or Ohmic contacts (in metal semi-conductor junctions) can be explained in terms of quantum mechanical tunneling, a counterpart of electromagnetic tunneling studied in this section. Also, quantum mechanical tunneling is fundamental to the operation of "scanning tunneling microscopes" used to image atoms and crystals.

- The transmission line analogy to solve a four-slab problem:

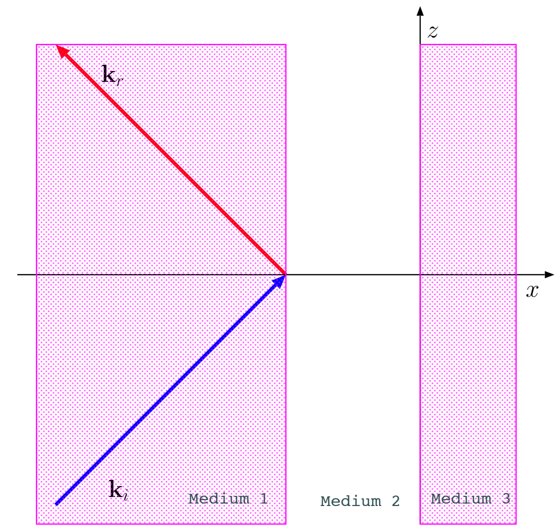


  - The relations shown on the diagram can be used to calculate the transmittance $1-|\Gamma_{10}|^2$ from region 0 to region 3 assuming that regions 1 and 2 are lossless.

Example 2: If in the above diagram region 3 is evanescent what would be the transmittance $1-|\Gamma_{10}|^2$?

Answer: In that case the transmittance should be zero (and reflectance unity)!

Tunneling at oblique incidence



- Our analysis of tunneling and frustrated-TIR at oblique incidence will amount to analyzing the three-slab geometry shown in the margin with interfaces at $x = -d$ and $x = 0$ surfaces separating media with refractive indeces n_1, n_2, and n_3, respectively.

- Assume that medium 1 has TE-polarized incident and reflecting electric fields superposing as

$$\hat{y}E_o(e^{-jk_1(\sin\theta_1 z+\cos\theta_1 x)} + Re^{-jk_1(\sin\theta_1 z-\cos\theta_1 x)});$$

  the field in medium 2 is

$$\hat{y}E_o(Pe^{-jk_2(\sin\theta_2 z+\cos\theta_2 x)} + Qe^{-jk_2(\sin\theta_2 z-\cos\theta_2 x)});$$

  and in medium 3 we have

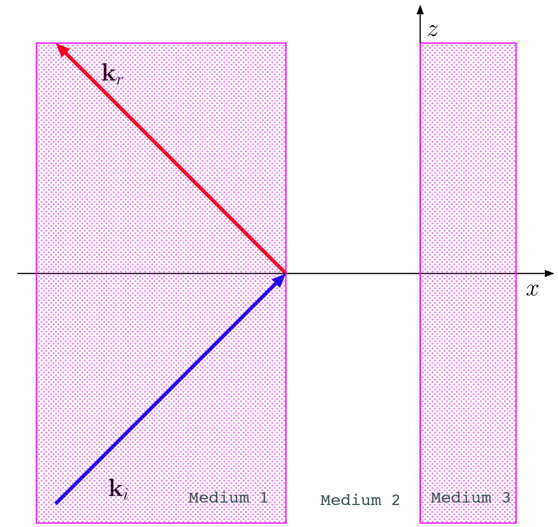
$$\hat{y}E_oTe^{-jk_3(\sin\theta_3 z+\cos\theta_3 x)}.$$

- In medium 1 the $\hat{z}$ component of total $\tilde{\mathbf{H}}$ is

$$\frac{E_o}{\eta_1}\cos\theta_1(e^{-jk_1(\sin\theta_1 z+\cos\theta_1 x)} - Re^{-jk_1(\sin\theta_1 z-\cos\theta_1 x)});$$

  in medium 2 we have

$$\frac{E_o}{\eta_2}\cos\theta_2(Pe^{-jk_2(\sin\theta_2 z+\cos\theta_2 x)} - Qe^{-jk_2(\sin\theta_2 z-\cos\theta_2 x)});$$



and in medium 3

$$\frac{E_o}{\eta_3}\cos\theta_3 T e^{-jk_3(\sin\theta_3 z+\cos\theta_3 x)}.$$

- BC's applied at $x=-d$ and $x=0$ require a "phase matching", that is

$$k_1\sin\theta_1 = k_2\sin\theta_2 = k_3\sin\theta_3,$$

leading to Snell's law relations

$$n_2\sin\theta_2 = n_1\sin\theta_1 \quad \text{and} \quad n_3\sin\theta_3 = n_1\sin\theta_1.$$

- Matching the tangential $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{H}}$ at $x=0$ boundary yields

$$P+Q=T \;\text{ with }\; \frac{P-Q}{\eta_2/\cos\theta_2} = \frac{T}{\eta_3/\cos\theta_3}$$

implies

$$\frac{P-Q}{\eta_2/\cos\theta_2} = \frac{P+Q}{\eta_3/\cos\theta_3} \quad \Rightarrow \quad \frac{Q}{P} = \frac{\eta_3/\cos\theta_3 - \eta_2/\cos\theta_2}{\eta_3/\cos\theta_3 + \eta_2/\cos\theta_3} \equiv \Gamma_{32}$$

defining an effective "load reflection" coefficient for this problem.

- Transverse field matching at $x=-d$ requires for $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{H}}$

$$e^{jk_1\cos\theta_1 d} + Re^{-jk_1\cos\theta_1 d} = Pe^{jk_2\cos\theta_2 d} + Qe^{-jk_2\cos\theta_2 d}$$

and

$$\frac{1}{\eta_1}\cos\theta_1(e^{jk_1\cos\theta_1 d} - Re^{-jk_1\cos\theta_1 d}) = \frac{1}{\eta_2}\cos\theta_2(Pe^{jk_2\cos\theta_2 d} - Qe^{-jk_2\cos\theta_2 d}),$$

respectively. Now define transverse field impedance

$$Z(-d) = \frac{E_y}{H_z} = \frac{Pe^{jk_2\cos\theta_2 d} + Qe^{-jk_2\cos\theta_2 d}}{\frac{Pe^{jk_2\cos\theta_2 d} - Qe^{-jk_2\cos\theta_2 d}}{\eta_2/\cos\theta_2}} = \frac{\eta_2}{\cos\theta_2}\frac{1+\Gamma_{32}e^{-j2k_2\cos\theta_2 d}}{1-\Gamma_{32}e^{-j2k_2\cos\theta_2 d}}$$

and

$$Z(-d) = \frac{E_y}{H_z} = \frac{e^{jk_1\cos\theta_1 d} + Re^{-jk_1\cos\theta_1 d}}{\frac{e^{jk_1\cos\theta_1 d} - Re^{-jk_1\cos\theta_1 d}}{\eta_1/\cos\theta_1}} = \frac{\eta_1}{\cos\theta_1}\frac{1+\Gamma_{21}}{1-\Gamma_{21}}$$

where

$$\Gamma_{21} \equiv \frac{Re^{-jk_1\cos\theta_1 d}}{e^{jk_1\cos\theta_1 d}}.$$

- Clearly $|\Gamma_{21}|^2$ is the *reflectance* (fraction of incident power density in the reflected wave) and $1-|\Gamma_{21}|^2$ is the *transmittance,* wherein

$$\Gamma_{21} = \frac{Z(-d) - \eta_1/\cos\theta_1}{Z(-d) - \eta_1/\cos\theta_1}.$$

These results suggest the use of transmission line analogy in terms of characteristic impedances $\eta_i/\cos\theta_i$ and $k_{xi} \equiv k_i\cos\theta_i$ in phase terms. Note that in evanescent regions $\cos\theta_i$ are purely imaginary. Also quarter-wave transformations can be used when $d\cos\theta_2 = \lambda_2/4$ and half-wave transformations when $d\cos\theta_2 = \lambda_2/2$.

- For the TM case, the use of $\eta_i \cos\theta_i$ in place of $\eta_i / \cos\theta_i$ in reflection coefficient and impedance calculations leads to the correct reflectance and transmittance values (same trick would also work in TE and TM Fresnel reflection coefficients in oblique reflections from a single interface as well as in guide impedance formulae where $\cos\theta$ is replaced by $\sqrt{1 - f_c^2/f^2}$).

  Here are the details:

  - Matching the tangential $\tilde{\mathbf{H}}$ and $\tilde{\mathbf{E}}$ at $x = 0$ boundary yields

    $$P{+}Q = T \ \text{ with } \ -\eta_2 \cos\theta_2 (P{-}Q) = -\eta_3 \cos\theta_3 T = -\eta_3 \cos\theta_3 (P{+}Q)$$

    implying

    $$\eta_2 \cos\theta_2 (P{-}Q) = \eta_3 \cos\theta_3 (P{+}Q) \quad \Rightarrow \quad \frac{Q}{P} = \frac{\eta_2 \cos\theta_2 - \eta_3 \cos\theta_3}{\eta_2 \cos\theta_2 + \eta_3 \cos\theta_3} \equiv -\Gamma_{32}$$

    defining an effective "load reflection" coefficient for this problem.

  - Transverse field matching at $x = -d$ requires, for $\tilde{\mathbf{H}}$ and $\tilde{\mathbf{E}}$,

    $$e^{jk_1 \cos\theta_1 d} + Re^{-jk_1 \cos\theta_1 d} = Pe^{jk_2 \cos\theta_2 d} + Qe^{-jk_2 \cos\theta_2 d}$$

    and

    $$-\eta_1 \cos\theta_1 (e^{jk_1 \cos\theta_1 d} {-} Re^{-jk_1 \cos\theta_1 d}) = -\eta_2 \cos\theta_2 (Pe^{jk_2 \cos\theta_2 d} {-} Qe^{-jk_2 \cos\theta_2 d}),$$

    respectively. Now define transverse field impedance

    $$Z(-d) = \frac{-E_z}{H_y} = \frac{\eta_2 \cos\theta_2 (Pe^{jk_2 \cos\theta_2 d} - Qe^{-jk_2 \cos\theta_2 d})}{Pe^{jk_2 \cos\theta_2 d} + Qe^{-jk_2 \cos\theta_2 d}} = \eta_2 \cos\theta_2 \frac{1 + \Gamma_{32} e^{-j2k_2 \cos\theta_2 d}}{1 - \Gamma_{32} e^{-j2k_2 \cos\theta_2 d}}$$

and

$$Z(-d) = \frac{-E_z}{H_y} = \frac{\eta_1 \cos\theta_1 (e^{jk_1 \cos\theta_1 d} - Re^{-jk_1 \cos\theta_1 d})}{e^{jk_1 \cos\theta_1 d} + Re^{-jk_1 \cos\theta_1 d}} = \eta_1 \cos\theta_1 \frac{1 + \Gamma_{21}}{1 - \Gamma_{21}}$$

where

$$\Gamma_{21} \equiv -\frac{Re^{-jk_1 \cos\theta_1 d}}{e^{jk_1 \cos\theta_1 d}}.$$

– The upshot is,

$$\Gamma_{21} = \frac{Z(-d) - \eta_1 \cos\theta_1}{Z(-d) + \eta_1 \cos\theta_1}$$

wherein we see the replacement of all $\eta_i / \cos\theta_i$ in TE-mode relations by $\eta_i \cos\theta_i$ to become the corresponding TM-mode relations.