21 Doppler — cont'd

Example 1: A space ship traveling between Earth and Moon is emitting a TEM wave at a radian frequency ω. The TEM wave reaching Earth is found to be oscillating with a radian frequency of $\omega^E = 2.999\,\pi\,10^9$ rad/sec while on the moon the wave frequency is measured as $\omega^M = 3.001\,\pi\,10^9$ rad/sec. (a) Determine ω, k and λ, where k and λ are the TEM wavenumber and wavelength, respectively, in the reference frame of the space ship. (b) Determine the velocity of the space ship in the Earth reference frame. Assume free-space propagation and that the distance between Earth and Moon is constant during the measurements.

Solution: (a) Clearly ω^E and ω^M can differ from ω by $\pm kv$ (in non-relativistic approximation) where v is the relative *speed* of the space shift with respect to Earth and Moon and k is the wavenumber in the space ship frame. Since $\omega^M > \omega^E$, we must have

$$\begin{aligned} \omega^M &= \omega + kv, \\ \omega^E &= \omega - kv. \end{aligned}$$

Hence,

$$\omega^M + \omega^E = 2\omega \;\Rightarrow\; \omega = \frac{\omega^M + \omega^E}{2} = 3\pi\,10^9\,\text{rad/s}.$$

It follows that

$$k = \frac{\omega}{c} = 10\pi \;\text{ and }\; \lambda = \frac{2\pi}{k} = 0.2\,\text{m}.$$

(b) Taking the difference of the above equations we also find that

$$\omega^M - \omega^E = 2kv \;\Rightarrow\; v = \frac{\omega^M - \omega^E}{2k} = \frac{2\pi\,10^6}{20\pi} = 10^5\,\text{m/s}.$$

Since ω^E is red-shifted with respect to ω, the space ship must be moving away from the Earth with the speed v.

Note that identifying the speed of the space shift in the Earth frame and its direction of motion is equivalent to identifying its *velocity*.

Finally, note that since $v \ll c$, our use of the non-relativistic formulae in above solution was well justified. You can also solve the same problem (with no approximations) starting with the relativistic relations

$$\omega^M = \omega\sqrt{\frac{1+v/c}{1-v/c}} \approx \omega + kv,$$
$$\omega^E = \omega\sqrt{\frac{1-v/c}{1+v/c}} \approx \omega - kv.$$

Try it and show to yourself that our earlier (inexact) solution was in fact very accurate!

Doppler radars (revisited):

- Doppler shift phenomenon is essential to the operation of Doppler radars such as weather radars or police radars for purposes of target motion measurements.

- In particular, the two-way Doppler shift equation

$$\omega'' = \omega\frac{1-\frac{v}{c}}{1+\frac{v}{c}} \approx \omega(1-\frac{v}{c})(1-\frac{v}{c}) \approx \omega(1-2\frac{v}{c}) = \omega - 2kv$$

  plays a major role because the reflected frequency ω'' from the moving target is compared to the incident frequency ω in order to estimate the target velocity v.

- In many applications the non-relativistic limit applies, i.e., $|v| \ll c$, in which case the target velocity v away from the radar antenna is obtained as

$$v = \frac{\omega - \omega''}{2k} = \frac{c}{2}\frac{\omega - \omega''}{\omega}.$$

  Also, v gives the component of the target velocity in the direction of propagation of the incident wave from the radar.

  - Transverse motion of the radar target with respect to the incident wave from the radar does not cause any Doppler shift.

Example 1: Police radars catch you when the magnitude of

$$v = \frac{\omega - \omega''}{2k} = \frac{c}{2}\frac{\omega - \omega''}{\omega} \quad \text{exceeds 70 mph — here}$$

ω is the radar transmission frequency while ω'' is the frequency of the wave that bounces off your car back to the antenna of the police radar.

Also v is the component of your car's vector velocity away from the radar.

When your car is approaching the radar $\omega'' > \omega$ and v is negative. Conversely, when your car is moving away from the radar $\omega'' < \omega$ and v is positive.

- Total reflection is not necessary for Doppler effect and the operation of Doppler radars. Partially reflected waves from a moving dielectric surface, or even scattered fields from atoms in a gas in motion re-radiating like tiny dipole antennas excited by the incident wave, will also produce Doppler shifted returns of the incident (transmitted) radar signal governed by the Doppler shift formulae above.

  - Meteorologists and atmospheric scientists routinely make wind measurements by bouncing EM waves from atmospheric atoms and ionospheric free electrons — also the research area of S. Franke, E. Kudeki, and J. Makela in the Remote Sensing Lab in our Dept.

- Engineers designing police radars and meteorologists building weather radars find themselves in strictly the non-relativistic domain $\frac{|v|}{c} \ll 1$. They will routinely think of the one-way Doppler shift formula

$$\omega' = \omega - kv$$

as the *time rate of change* of a plane wave phase

$$\omega t - kx$$

evaluated at the location

$$x = vt$$

of a moving observer.

In a broad sense, frequency of a wave in any reference frame is the time rate of change of the wave phase, and observers in relative motion naturally detect different rates in a wave field.

Example 2: A police radar with an operation frequency of $f = 300$ MHz is located at the origin $(x, y, z) = (0, 0, 0)$. A car with the trajectory

$$(x, y, z) = (50t, 50, 0)$$

is passing by, where the coordinates are given in meter units ant time t is measured in seconds.

(a) Determine the vector velocity of the car.

(b) Determine the frequency ω' of the radar signal in the reference frame of the car, by determining the rate of change of the signal phase detected by an antenna connected to the car.

(c) Determine the two-way shifted radar frequency ω'' by considering the rate of change of the phase delay of the reflected signal.

Solution: (a) The vector velocity of the car is

$$\mathbf{v} = \frac{\partial \mathbf{r}}{\partial t} = \frac{\partial}{\partial t}(50t, 50, 0) = (50, 0, 0) = 50\hat{x}\,\frac{\text{m}}{\text{s}}.$$

(b) The radial distance from the radar to the car is given by

$$r = \sqrt{(50t)^2 + 50^2}.$$

Therefore, the spherical wave phasor of the radar signal at the location of the car is proportional to

$$e^{-jkr} = e^{-jk\sqrt{(50t)^2+50^2}},$$

where $k = \omega/c = 2\pi$ rad/m. Therefore, the field at the location of the car varies with time in proportion to the real part of

$$e^{-jkr}e^{j\omega t} = e^{j(\omega t - k\sqrt{(50t)^2+50^2})}.$$

Thus, the phase of the signal detected by the antenna connected to the car is

$$\Phi(t) = \omega t - k\sqrt{(50t)^2 + 50^2}.$$

Finally, the frequency of the incident radar signal detected in the car frame is

$$\omega' = \frac{\partial \Phi}{\partial t} = \omega - k\frac{\frac{1}{2}(2(50t)50 + 0)}{\sqrt{(50t)^2 + 50^2}} = \omega - k50\frac{50t}{\sqrt{(50t)^2 + 50^2}} = \omega - k50\frac{t}{\sqrt{t^2 + 1}}.$$

(c) The field reflected from the moving car corresponds to the real part of

$$e^{-j2kr}e^{j\omega t} = e^{j(\omega t - 2k\sqrt{(50t)^2 + 50^2})}$$

since the phase delay occurs twice over the distance r. This leads to the two-way Doppler shifted radar frequency formula

$$\omega'' = \omega - k100\frac{t}{\sqrt{t^2 + 1}}.$$

Note that $\omega' = \omega'' = \omega$ at $t = 0$ when the car motion is transverse to the propagation direction of the incident radar wave.

Also note that the two-way Doppler shift

$$\omega'' - \omega = -k100\frac{t}{\sqrt{t^2 + 1}}$$

maximizes in magnitude at

$$|\omega'' - \omega| = 100k = 200\pi\,\frac{rad}{\mathrm{s}} \quad \Leftrightarrow \quad 100\,\mathrm{Hz}$$

for $t \ll -1$ s and $t \gg 1$ s when the car's motion is nearly collinear with the propagation direction of the incident wave from the radar.