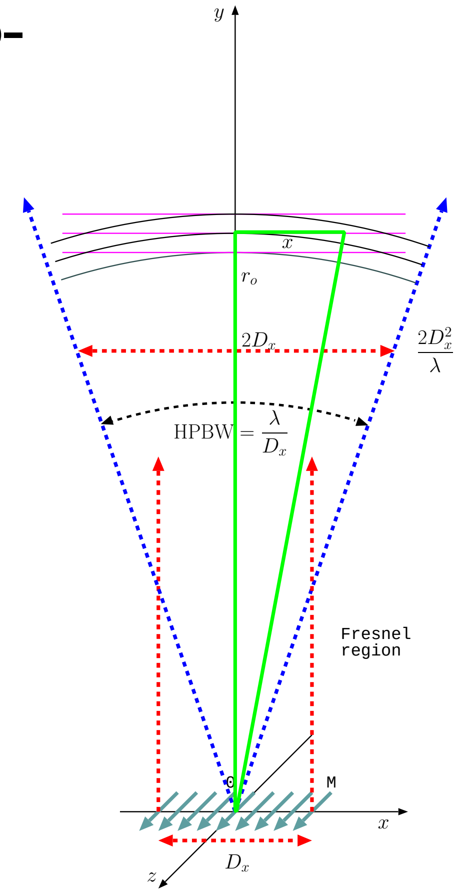
15 Plane-wave form of Maxwell's equations, propagation in arbitrary direction



Having seen how EM waves are generated by radiation sources and how spherical TEM waves develop a "planar" character over increasingly large regions as they propagate away from their sources, it is time to shift our attention to *propagation* and *guidance phenomena* using a plane-wave formalism.

Perhaps the most "practical" rationalization of this switch from spherical to plane-wave emphasis is that waves produced by compact sources invariably "look" planar at the scales of practical receiving systems (that will study near the end of this course) situated afar.

- We wish to study wave solutions of Maxwell's equations exhibiting the planar phasor form

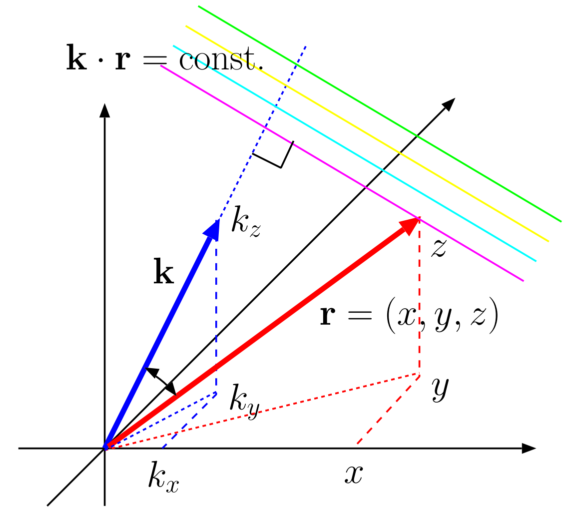
  $$\tilde{\mathbf{E}} = \mathbf{E}_o e^{-j\mathbf{k}\cdot\mathbf{r}} = \hat{e} E_o e^{-j\mathbf{k}\cdot\mathbf{r}}$$

  and time-domain variations

  $$\begin{aligned} \text{Re}\{\tilde{\mathbf{E}} e^{j\omega t}\} &= \text{Re}\{\mathbf{E}_o e^{j(\omega t - \mathbf{k}\cdot\mathbf{r})}\} \\ &= \hat{e}|E_o|\cos(\omega t - \mathbf{k}\cdot\mathbf{r} + \angle E_o) \end{aligned}$$

  where **wave vector k** is to be found in compliance with ω and Maxwell's equations according to some specific "dispersion relation" including the details of the propagation medium.

– For simplicity, the above phasor has been declared to be linearly polarized. Circular or elliptic polarized wave fields can be constructed later on via superposition methods.



- Linearly polarized wave field phasor above can be expanded as

$$\tilde{\mathbf{E}} = \mathbf{E}_o e^{-j\mathbf{k}\cdot\mathbf{r}} = \mathbf{E}_o e^{-j(k_x x + k_y y + k_z z)}$$

assuming a wave vector

$$\mathbf{k} = (k_x, k_y, k_z) = \hat{x}k_x + \hat{y}k_y + \hat{z}k_z$$

expressed in terms of its projections (k_x, k_y, k_z) along the Cartesian coordinate axes (x, y, z).

- A special case we are familiar with is

$$k_x = k_y = 0,\ k_z > 0,\ \text{when } \mathbf{k} = k_z\hat{z} = k\hat{z} \text{ and } e^{-j\mathbf{k}\cdot\mathbf{r}} = e^{-jkz}$$

as in plane TEM waves travelling in $+z$ direction having a

$$\text{wavelength } \lambda = \frac{2\pi}{k} \text{ and propagation speed } v_p = \frac{\omega}{k}.$$

– Likewise, the case

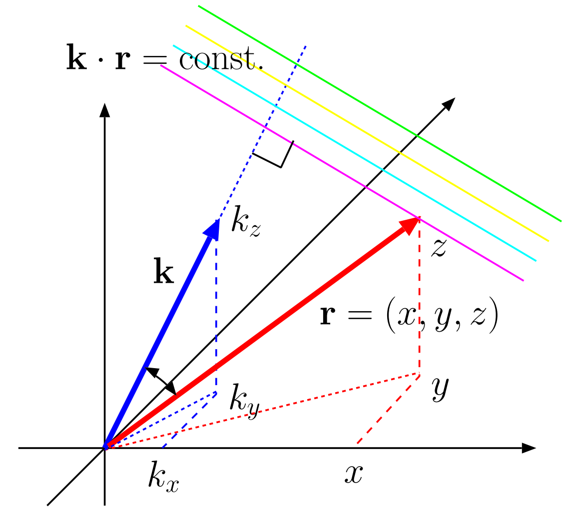
$$k_y = k_z = 0\,, k_x > 0,\ \text{when } \mathbf{k} = k_x\hat{x} = k\hat{x} \text{ and } e^{-j\mathbf{k}\cdot\mathbf{r}} = e^{-jkx}$$

corresponds to plane TEM waves travelling in $+x$ direction with the same wavelength and propagation speed.

- The general case with non-zero components (k_x, k_y, k_z) corresponds to a plane wave propagating in the direction of unit vector

$$\hat{k} \equiv \frac{\mathbf{k}}{k} = \frac{(k_x, k_y, k_z)}{k} \text{ where } k \equiv |\mathbf{k}| = \sqrt{k_x^2 + k_y^2 + k_z^2} = \frac{2\pi}{\lambda}$$

and also having the same wavelength and propagation speed as above. Wavelength λ now describes the shift invariance of the wave field in spatial $\hat{k}$ direction, i.e., the propagation direction.



Example 1: A plane wave electric field phasor is specified as

$$\tilde{\mathbf{E}} = \hat{z} e^{-j(3\pi x - 4\pi y)} \frac{\text{V}}{\text{m}}.$$

Determine the propagation direction $\hat{k}$, wavenumber $k = |\mathbf{k}|$, wavelength $\lambda = \frac{2\pi}{k}$ and wave frequency $f = \frac{\omega}{2\pi}$ assuming a propagation speed $c = 3 \times 10^8$ m/s.

Solution: Contrasting $\tilde{\mathbf{E}}$ with $e^{-j(k_x x + k_y y + k_z z)}$, we note that

$$k_x = 3\pi \frac{\text{rad}}{\text{m}}, \;\; k_y = -4\pi \frac{\text{rad}}{\text{m}}, \;\; k_z = 0.$$

Hence, wave vector

$$\mathbf{k} = \hat{x} k_x + \hat{y} k_y + \hat{z} k_z = 3\pi\hat{x} - 4\pi\hat{y} \frac{\text{rad}}{\text{m}},$$

and wave number

$$k = |\mathbf{k}| = \sqrt{k_x^2 + k_y^2 + k_z^2} = \sqrt{(3\pi)^2 + (4\pi)^2 + 0^2} = \sqrt{25\pi^2} = 5\pi \frac{\text{rad}}{\text{m}}.$$

The propagation direction is specified by the unit vector

$$\hat{k} = \frac{\mathbf{k}}{k} = \frac{3\pi\hat{x} - 4\pi\hat{y}}{5\pi} = 0.6\hat{x} - 0.8\hat{y}.$$

The wavelength is

$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{5\pi} = 0.4\,\text{m}.$$

Since

$$c = v_p = \frac{\omega}{k}$$

in general, it follows that

$$\omega = kc = 5\pi \times 3 \times 10^8 = 2\pi \times 7.5 \times 10^8\,\frac{\text{rad}}{\text{s}}$$

and

$$f = \frac{\omega}{2\pi} = 750 \times 10^6\,\text{Hz} = 750\,\text{MHz}.$$

- Based on what we learned in ECE 329, we recognize that the wave analyzed in Example 1 must have been propagating in free space.
- What are the constraints on wave vector $\mathbf{k}$ for plane waves propagating in arbitrary media?

To answer the above question, we will return to macroscopic-form Maxwell's equations written in phasor form (see margin) and examine under which conditions phasor solutions

$$\propto e^{-j\mathbf{k}\cdot\mathbf{r}}$$

can be applicable for all the field quantities in the absence of source currents $\tilde{\mathbf{J}}$ and their accompanying $\tilde{\rho}$.

- First, we note that in view of relation

  $$\tilde{\mathbf{D}} = \epsilon\tilde{\mathbf{E}},$$

  we can have plane-wave solutions of the form

  $$\tilde{\mathbf{D}} = \mathbf{D}_o e^{-j\mathbf{k}\cdot\mathbf{r}} \text{ and } \tilde{\mathbf{E}} = \mathbf{E}_o e^{-j\mathbf{k}\cdot\mathbf{r}}$$

  if and only if ϵ does not depend on position $\mathbf{r}$ (why?).

- Likewise, relation

  $$\tilde{\mathbf{B}} = \mu\tilde{\mathbf{H}},$$

  implies plane-wave solutions

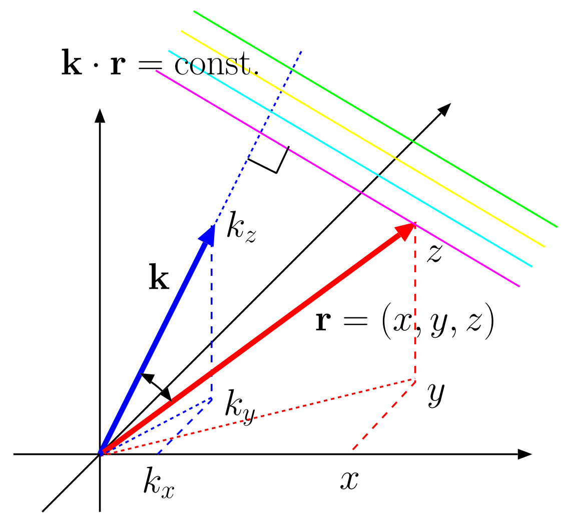
  $$\tilde{\mathbf{B}} = \mathbf{B}_o e^{-j\mathbf{k}\cdot\mathbf{r}} \text{ and } \tilde{\mathbf{H}} = \mathbf{H}_o e^{-j\mathbf{k}\cdot\mathbf{r}}$$

  if and only if μ does not depend on position $\mathbf{r}$ (why?).

$$\begin{aligned} \nabla\cdot\tilde{\mathbf{D}} &= \tilde{\rho} \\ \nabla\cdot\tilde{\mathbf{B}} &= 0 \\ \nabla\times\tilde{\mathbf{E}} &= -j\omega\tilde{\mathbf{B}} \\ \nabla\times\tilde{\mathbf{H}} &= \tilde{\mathbf{J}} + j\omega\tilde{\mathbf{D}} \end{aligned}$$

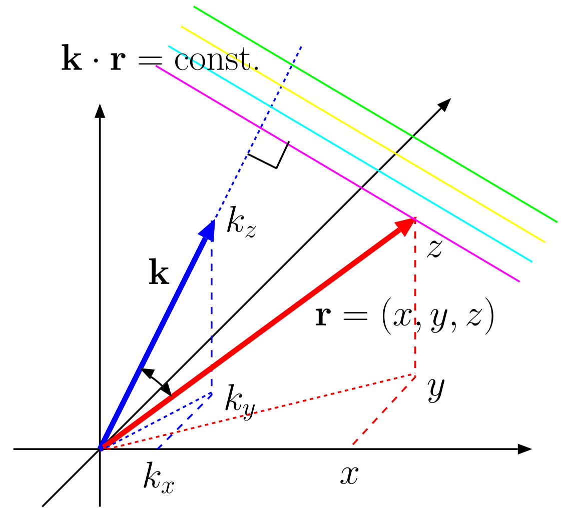
where (constitutive relations)

$$\begin{aligned} \tilde{\mathbf{D}} &= \epsilon\tilde{\mathbf{E}} \\ \tilde{\mathbf{B}} &= \mu\tilde{\mathbf{H}} \\ \tilde{\mathbf{J}}_c &= \sigma\tilde{\mathbf{E}}. \end{aligned}$$



- In a homogeneous region where ϵ, μ, and σ are, by definition, independent of $\mathbf{r}$, plane-wave solutions of phasor-form Maxwell's equations given in the margin become possible provided that

$$\begin{aligned} -j\mathbf{k}\cdot\tilde{\mathbf{D}} &= \tilde{\rho} \\ -j\mathbf{k}\cdot\tilde{\mathbf{B}} &= 0 \\ -j\mathbf{k}\times\tilde{\mathbf{E}} &= -j\omega\tilde{\mathbf{B}} \\ -j\mathbf{k}\times\tilde{\mathbf{H}} &= \tilde{\mathbf{J}}+j\omega\tilde{\mathbf{D}}. \end{aligned}$$



We have obtained these vector-algebraic relations from phasor-form Maxwell's equations in the margin after replacing the vector-differential operator ∇ by the vector-algebraic operator $-j\mathbf{k}$.

The justification of this simple procedure is as follows:

If

$$\tilde{\mathbf{D}}=\mathbf{D}_o e^{-j\mathbf{k}\cdot\mathbf{r}}=\mathbf{D}_o e^{-j(k_x x+k_y y+k_z z)}=(D_{xo},D_{yo},D_{zo})e^{-j(k_x x+k_y y+k_z z)}$$

then

$$\begin{aligned} \nabla\cdot\tilde{\mathbf{D}} &= (\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial y})\cdot(D_{xo}e^{-j(k_x x+k_y y+k_z z)},D_{yo}e^{-j(k_x x+k_y y+k_z z)},D_{zo}e^{-j(k_x x+k_y y+k_z z)}) \\ &= -jk_x D_{xo}e^{-j(k_x x+k_y y+k_z z)}-jk_y D_{yo}e^{-j(k_x x+k_y y+k_z z)}-jk_z D_{zo}e^{-j(k_x x+k_y y+k_z z)} \\ &= -j(k_x,k_y,k_z)\cdot(D_{xo},D_{yo},D_{zo})e^{-j(k_x x+k_y y+k_z z)}=-j\mathbf{k}\cdot\tilde{\mathbf{D}}. \end{aligned}$$

Likewise, if

$$\tilde{\mathbf{E}}=\mathbf{E}_o e^{-j\mathbf{k}\cdot\mathbf{r}}=\mathbf{E}_o e^{-j(k_x x+k_y y+k_z z)}=(E_{xo},E_{yo},E_{zo})e^{-j(k_x x+k_y y+k_z z)}$$

then

$$
\begin{aligned}
\nabla \times \tilde{\mathbf{E}} &= (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial y}) \times (E_{xo}e^{-j(k_x x + k_y y + k_z z)}, E_{yo}e^{-j(k_x x + k_y y + k_z z)}, E_{zo}e^{-j(k_x x + k_y y + k_z z)}) \\
&= (-jk_x, -jk_y, -jk_z) \times (E_{xo}e^{-j(k_x x + k_y y + k_z z)}, E_{yo}e^{-j(k_x x + k_y y + k_z z)}, E_{zo}e^{-j(k_x x + k_y y + k_z z)}) \\
&= -j\mathbf{k} \times \tilde{\mathbf{E}}.
\end{aligned}
$$

The vector-algebraic relations above, repeated in the margin (after canceling out some common terms), are known as plane-wave form of Maxwell's equations.

Plane-wave form of Maxwell's equations:

$$
\begin{aligned}
-j\mathbf{k} \cdot \tilde{\mathbf{D}} &= \tilde{\rho} \\
\mathbf{k} \cdot \tilde{\mathbf{B}} &= 0 \\
\mathbf{k} \times \tilde{\mathbf{E}} &= \omega \tilde{\mathbf{B}} \\
-j\mathbf{k} \times \tilde{\mathbf{H}} &= \tilde{\mathbf{J}} + j\omega \tilde{\mathbf{D}}.
\end{aligned}
$$

- Plane-wave form ME in the margin provide us with the constraints such plane waves satisfy in various types of propagation media categorized according to ϵ, μ, and σ.

- Focusing first on the case $\tilde{\rho} = \tilde{\mathbf{J}} = 0$ and $\sigma = 0$ (source free and non-conducting), the equations simplify as

$$
\begin{aligned}
\mathbf{k} \cdot \tilde{\mathbf{D}} &= 0 \\
\mathbf{k} \cdot \tilde{\mathbf{B}} &= 0 \\
\mathbf{k} \times \tilde{\mathbf{E}} &= \omega \tilde{\mathbf{B}} \\
-\mathbf{k} \times \tilde{\mathbf{H}} &= \omega \tilde{\mathbf{D}}.
\end{aligned}
$$

  The first two constraints tell us that wave vector $\mathbf{k}$ is necessarily orthogonal to both $\tilde{\mathbf{D}} = \epsilon \tilde{\mathbf{E}}$ and $\tilde{\mathbf{B}} = \mu \tilde{\mathbf{H}}$.

  - Hence, the plane waves satisfying the above equations will be TEM.

- Cross-multiplying the third equation with $\mathbf{k}$ and substituting from the fourth equation we get

$$\mathbf{k} \times (\mathbf{k} \times \tilde{\mathbf{E}}) = \omega\mu \mathbf{k} \times \tilde{\mathbf{H}} = \omega\mu(-\omega\tilde{\mathbf{D}}) = -\mu\epsilon\omega^2\tilde{\mathbf{E}}.$$

But we also have

$$\mathbf{k} \times (\mathbf{k} \times \tilde{\mathbf{E}}) = -(\mathbf{k} \cdot \mathbf{k})\tilde{\mathbf{E}}$$

since vectors $\mathbf{k}$ and $\tilde{\mathbf{E}}$ are perpendicular as shown in the margin — cross-multiplying $\tilde{\mathbf{E}}$ twice by $\mathbf{k} = k\hat{k}$ produces $-\tilde{\mathbf{E}}$ times $k^2 \equiv \mathbf{k} \cdot \mathbf{k}$!

  - The above lines are compatible if and only if

$$\mathbf{k} \cdot \mathbf{k} = \omega^2\mu\epsilon \quad \Rightarrow \quad \hat{k} \cdot \hat{k} = 1 \text{ and } k = \omega\sqrt{\mu\epsilon},$$

which is the **dispersion relation** of TEM plane-wave solutions of Maxwell' equations

$$\propto e^{-j\omega\sqrt{\mu\epsilon}\hat{k}\cdot\mathbf{r}}$$

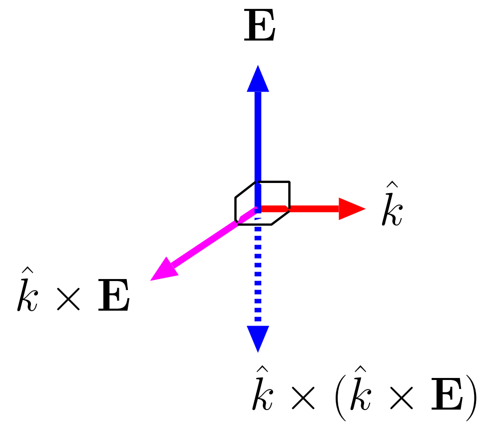
with

$$\hat{k} \cdot \tilde{\mathbf{E}} = 0 \quad \text{and} \quad \hat{k} \cdot \tilde{\mathbf{H}} = 0.$$

as well as (according to the last two equations in the margin)

$$\tilde{\mathbf{H}} = \frac{\hat{k} \times \tilde{\mathbf{E}}}{\eta} \quad \text{and} \quad \tilde{\mathbf{E}} = \eta\tilde{\mathbf{H}} \times \hat{k} \quad \text{with} \quad \eta = \sqrt{\frac{\mu}{\epsilon}}.$$

- TEM plane wave solutions obtained above correspond to *undamped uniform* plane waves when the **wavevector k** obeying the **dispersion relation** $\mathbf{k} \cdot \mathbf{k} = \omega^2\mu\epsilon$ is real valued.



Also, the vector identity

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{C}\cdot\mathbf{A})\mathbf{B} - (\mathbf{B}\cdot\mathbf{A})\mathbf{C}$$

leads to the same result.

Plane-wave form of Maxwell's equations:

$$\begin{aligned} \mathbf{k} \cdot \tilde{\mathbf{D}} &= 0 \\ \mathbf{k} \cdot \tilde{\mathbf{B}} &= 0 \\ \mathbf{k} \times \tilde{\mathbf{E}} &= \omega\mu\tilde{\mathbf{H}} \\ -\mathbf{k} \times \tilde{\mathbf{H}} &= \omega\epsilon\tilde{\mathbf{E}}. \end{aligned}$$

- Same results also describe *damped* plane waves and/or *non-uniform* plane waves with *complex valued* $\mathbf{k}$:
  - Damped waves: if $\hat{k}$ is real but $k = \omega\sqrt{\mu\epsilon}$ is complex valued with a negative imsaginary part - e.g., in Ohmic conductors
  - Non uniform waves: if $\hat{k}$, obeying $\hat{k} \cdot \hat{k} = 1$ is a complex valued unit vector - e.g., with *surface waves*, *evanescent waves* ... to be studied over the next few weeks

- Example: Non-uniform plane waves with real valued $\mathbf{k} \cdot \mathbf{k}$
  - Consider $\mathbf{k} \cdot \mathbf{k} = \omega^2 \mu_o \epsilon_o$ where the right hand side is real valued and equal to the square of ω/c.
  - Let $\mathbf{k} = \mathbf{k}_r + j\mathbf{k}_i$ where $\mathbf{k_r}$ and $\mathbf{k}_i$ are real valued.
  - Then $\mathbf{k} \cdot \mathbf{k} = (\mathbf{k}_r \cdot \mathbf{k}_r - \mathbf{k}_i \cdot \mathbf{k}_i) + j2\mathbf{k}_r \cdot \mathbf{k}_i = \omega^2 \mu_o \epsilon_o$ leading to the constraints

$$\mathbf{k}_r \cdot \mathbf{k}_r - \mathbf{k}_i \cdot \mathbf{k}_i = \omega^2 \mu_o \epsilon_o$$
$$\mathbf{k}_r \cdot \mathbf{k}_i = 0.$$

  - For instance $\mathbf{k} = (k_x, k_y, k_z) = (2\pi, 0, -j\pi)$ will comply with these constraints with $\mathbf{k}_r = (2\pi, 0, 0)$ and $\mathbf{k}_i = (0, 0, -\pi)$ and $\omega^2 \mu_o \epsilon_o = 3\pi^2$, describing a non-uniform plane wave with a phasor

$$e^{-j\mathbf{k}\cdot\mathbf{r}} = e^{-j(2\pi x - j\pi z)} = e^{-j2\pi x} e^{-\pi z}$$

that *propagates* in x direction with a wavelength of $\lambda = 2\pi/k_x = 1$ m and *decays* in z direction ... namely a "surface wave" propagating along, say, $z = 0$ surface.

- Translating the wave pahsor back to time domain, we see that it will be described as

$$\mathrm{Re}\{e^{-j\mathbf{k}\cdot\mathbf{r}}e^{j\omega t}\} = \mathrm{Re}\{e^{-j2\pi x}e^{-\pi z}e^{j\omega t}\} = e^{-\pi z}\cos(\omega t - 2\pi x).$$