11 Beam pattern, wave interference

In this lecture we will see how antenna beams can be "patterned" by using interference effects of fields radiated by multiple dipoles or dipole-like elements.

- Let's recall that the **antenna beam** is the shape of the antenna gain function $G(\theta, \phi)$ that can be depicted as a surface plot in 3D.

  Also

  $$D = G(\theta, \phi)_{max} = \frac{4\pi}{\Omega_o} \quad \text{and} \quad \Omega_o = \int d\Omega \frac{G(\theta, \phi)}{G(\theta, \phi)_{max}} = \int d\Omega \frac{|\langle \mathbf{E} \times \mathbf{H} \rangle|}{|\langle \mathbf{E} \times \mathbf{H} \rangle|_{max}}$$

  as well as

  $$|\langle \mathbf{E} \times \mathbf{H} \rangle| = \frac{|\tilde{E}_\theta|^2}{2\eta_o} \quad \text{with} \quad \tilde{E}_\theta = j\eta_o I_o k\ell \sin\theta \frac{e^{-jkr}}{4\pi r}$$

  for $\hat{z}$-polarized antennas and elements.

  - With $\ell = L/2$ the above equations would represent a short dipole.
  - An antenna system constructed by an *array* of such dipoles would also be represented by the same equations, but with a different $\ell = \ell(\theta, \phi)$ (to be determined).

- The design and analysis of multi-element or multi-dipole arrays are facilitated by the **linearity** of wave solutions of Maxwell's equations:
  - If radiators $\tilde{\mathbf{J}}_1$ and $\tilde{\mathbf{J}}_2$ produce radiated wave solutions $\tilde{\mathbf{E}}_1$ and $\tilde{\mathbf{E}}_2$, respectively, then a radiator $\alpha\tilde{\mathbf{J}}_1 + \beta\tilde{\mathbf{J}}_2$ would produce a wave solution $\alpha\tilde{\mathbf{E}}_1 + \beta\tilde{\mathbf{E}}_2$ with arbitrary (complex) weights α and β.
  - By induction, the above principle of superposition can be extended to n elements.

If

$$\tilde{\mathbf{J}}_1 \rightarrow \boxed{\mathbf{ME}} \rightarrow \tilde{\mathbf{E}}_1$$

and

$$\tilde{\mathbf{J}}_2 \rightarrow \boxed{\mathbf{ME}} \rightarrow \tilde{\mathbf{E}}_2$$

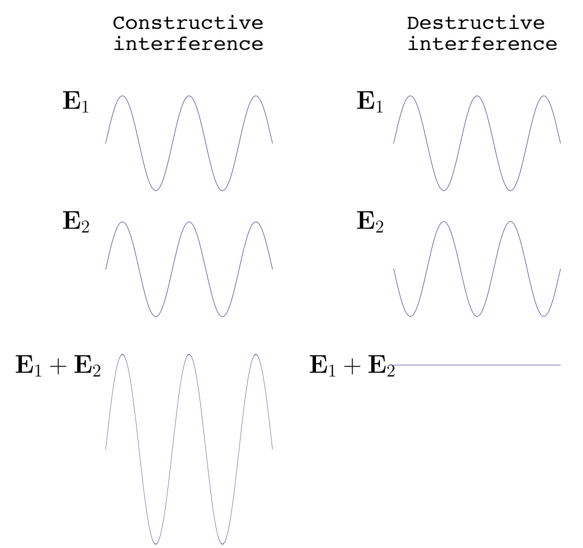
then

$$\alpha\tilde{\mathbf{J}}_1 + \beta\tilde{\mathbf{J}}_2 \rightarrow \boxed{\mathbf{ME}} \rightarrow \alpha\tilde{\mathbf{E}}_1 + \beta\tilde{\mathbf{E}}_2$$

Note that this superposition principle applies at the level of fields rather than power. This is similar to superposition principle applying at the level of voltage and currents in circuit analysis.

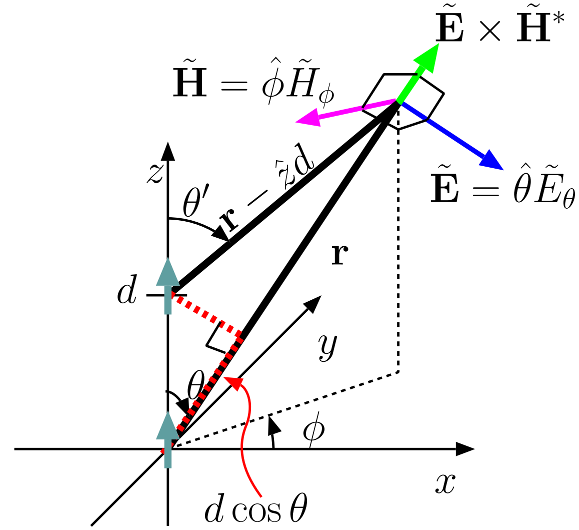
Superposition of wave fields can produce resultant wave fields with enhanced or reduced wave amplitudes as a consequence of **interference** effects.

- A **constructive interference** occurs at locations where the waves being superposed are "in phase", meaning that the phasors representing the wave fields are complex numbers having the same angle — i.e., $\angle\tilde{\mathbf{E}}_2 = \angle\tilde{\mathbf{E}}_1$.
- A **destructive interference** occurs where the waves being superposed are "out of phase", meaning that the phasors representing the wave fields are complex numbers having an angle difference of $\pm 180°$ — i.e., $\angle\tilde{\mathbf{E}}_2 = \angle\tilde{\mathbf{E}}_1 \pm 180°$.



With a judicious choice of the locations and relative amplitudes of the radiators $\tilde{\mathbf{J}}_1$ and $\tilde{\mathbf{J}}_2$, it is possible to arrange for $\alpha\tilde{\mathbf{E}}_1 + \beta\tilde{\mathbf{E}}_2$ to exhibit constructive interference in desired beam directions — that is the essence of antenna beam design and designing high directivity antenna systems.

- One final detail before showing some examples: the calculation of the superposed wave fields is considerably simplified at distances r to the source elements that far exceed the largest distance separating the source elements.



Example 1: Two $\hat{z}$ polarized dipole antennas with equal input currents I_o are located at $(0,0,0)$ and $(0,0,d)$. Find the phasor expression $\tilde{\mathbf{E}}(\mathbf{r})$ representing the superposition of the fields radiated by each dipole individually. What are the maximum and minimum values of the field intensity $|\tilde{\mathbf{E}}(\mathbf{r})|$ as compared to intensity $|\tilde{\mathbf{E}}_1(\mathbf{r})|$ of the field due to the dipole at the origin?

Solution: First, the dipole at $(x,y,z) = (0,0,0)$ has a wave field phasor

$$\tilde{\mathbf{E}}_1(\mathbf{r}) = j\eta_o I_o k\ell \sin\theta \frac{e^{-jkr}}{4\pi r}\hat{\theta} = j\eta_o I_o k\ell \sin\theta \frac{e^{-jk|\mathbf{r}|}}{4\pi|\mathbf{r}|}\hat{\theta}.$$

The field phasor of the second dipole at $(x,y,z) = (0,0,d)$ is a shifted counterpart of $\tilde{\mathbf{E}}_1$, namely

$$\tilde{\mathbf{E}}_2(\mathbf{r}) = \tilde{\mathbf{E}}_1(\mathbf{r} - \hat{z}d) = j\eta_o I_o k\ell \sin\theta' \frac{e^{-jk|\mathbf{r}-\hat{z}d|}}{4\pi|\mathbf{r} - \hat{z}d|}\hat{\theta}',$$

where the angle θ' is the angle between vectors $\hat{z}$ and $\mathbf{r}-\hat{z}d$ (see margin) such that

$$\hat{z}.\frac{(\mathbf{r}-\hat{z}d)}{|\mathbf{r}-\hat{z}d|}=\cos\theta'.$$

When both dipoles are "on", the total electric field phasor is

$$\tilde{\mathbf{E}}(\mathbf{r})=\tilde{\mathbf{E}}_1(\mathbf{r})+\tilde{\mathbf{E}}_2(\mathbf{r})=j\eta_o I_o k\ell[\sin\theta\frac{e^{-jk|\mathbf{r}|}}{4\pi|\mathbf{r}|}\hat{\theta}+\sin\theta'\frac{e^{-jk|\mathbf{r}-\hat{z}d|}}{4\pi|\mathbf{r}-\hat{z}d|}\hat{\theta}'].$$

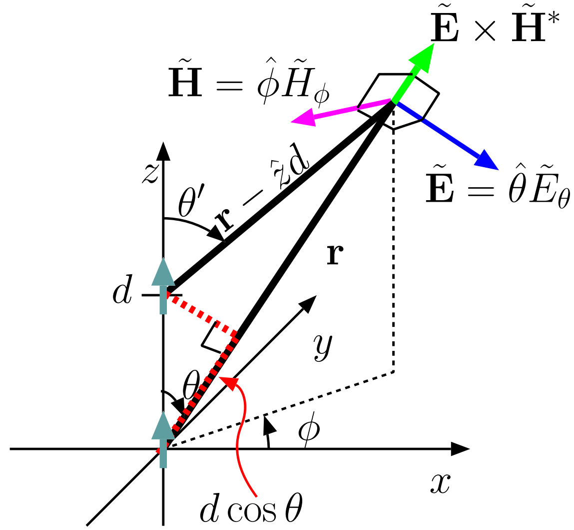
This superposition field phasor can also be expressed more compactly as

$$\tilde{\mathbf{E}}(\mathbf{r})=j\eta_o I_o k\ell\sin\theta\frac{e^{-jk|\mathbf{r}|}}{4\pi|\mathbf{r}|}[\hat{\theta}+\frac{\sin\theta'}{\sin\theta}\frac{|\mathbf{r}|}{|\mathbf{r}-\hat{z}d|}\frac{e^{-jk|\mathbf{r}-\hat{z}d|}}{e^{-jk|\mathbf{r}|}}\hat{\theta}'],$$

from which it follows that

$$|\tilde{\mathbf{E}}(\mathbf{r})|=|\tilde{\mathbf{E}}_1(\mathbf{r})||\hat{\theta}+\frac{\sin\theta'}{\sin\theta}\frac{|\mathbf{r}|}{|\mathbf{r}-\hat{z}d|}\frac{e^{-jk|\mathbf{r}-\hat{z}d|}}{e^{-jk|\mathbf{r}|}}\hat{\theta}'|.$$

From this result it is evident that $|\tilde{\mathbf{E}}(\mathbf{r})|$ can be at most twice $|\tilde{\mathbf{E}}_1(\mathbf{r})|$ when the primed term on the right approaches $\hat{\theta}$ (constructive interference), but it can also vanish when the primed term on the right approaches $-\hat{\theta}$ (destructive interference).



Example 2: Simplify the superposition field

$$\tilde{\mathbf{E}}(\mathbf{r}) = \tilde{\mathbf{E}}_1(\mathbf{r}) + \tilde{\mathbf{E}}_2(\mathbf{r}) = j\eta_o I_o k\ell \sin\theta \frac{e^{-jk|\mathbf{r}|}}{4\pi|\mathbf{r}|}[\hat{\theta} + \frac{\sin\theta'}{\sin\theta}\frac{|\mathbf{r}|}{|\mathbf{r}-\hat{z}d|}\frac{e^{-jk|\mathbf{r}-\hat{z}d|}}{e^{-jk|\mathbf{r}|}}\hat{\theta}']$$

from Example 1 by making **paraxial approximation** in the expansion of $|\mathbf{r}-\hat{z}d|$ in relation to $|\mathbf{r}|$. From the simplified expression, find the *effective length* ℓ_{eff} of the two element antenna array of short dipoles by forcing $\tilde{\mathbf{E}}(\mathbf{r})$ to have the standard form of a $\hat{z}$-polarized radiation field.

Solution: Making **paraxial approximation** in the expansion of $|\mathbf{r}-\hat{z}d|$ in relation to $|\mathbf{r}|$ amounts to having $|\mathbf{r}| = r \gg d$ so that vectors $\mathbf{r}$ and $\mathbf{r}-\hat{z}d$ can be regarded as being parallel — under that condition we can use $\theta' = \theta$, $\hat{\theta}' = \hat{\theta}$, and

$$|\mathbf{r}-\hat{z}d| = |\mathbf{r}| - d\cos\theta.$$

Then, the total field phasor simplifies as

$$\begin{aligned}\tilde{\mathbf{E}}(\mathbf{r}) &= j\eta_o I_o k\ell \sin\theta \frac{e^{-jk|\mathbf{r}|}}{4\pi|\mathbf{r}|}\hat{\theta}[1 + \frac{|\mathbf{r}|}{|\mathbf{r}| - d\cos\theta}\frac{e^{-jk(|\mathbf{r}|-d\cos\theta)|}}{e^{-jk|\mathbf{r}|}}] \\ &\approx \tilde{\mathbf{E}}_1(\mathbf{r})[1 + e^{jkd\cos\theta}].\end{aligned}$$

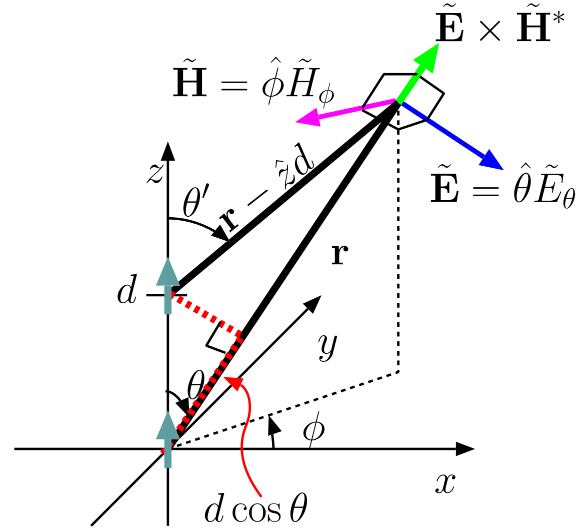
Alternatively,

$$\tilde{\mathbf{E}}(\mathbf{r}) = j\eta_o I_o k \underbrace{\ell[1 + e^{jkd\cos\theta}]}_{\ell_{eff}} \sin\theta \frac{e^{-jk|\mathbf{r}|}}{4\pi|\mathbf{r}|}\hat{\theta},$$

from which we have

$$\ell_{eff} = \ell[1 + e^{jkd\cos\theta}]$$

for the effective length of the array in terms of the effective length $\ell = \frac{L}{2}$ of the short-dipole array element.



Example 3: For the two-element antenna array of short dipoles examined in Examples 1 and 2 with field phasor

$$\tilde{\mathbf{E}}(\mathbf{r}) = j\eta_o I_o k \underbrace{\ell[1 + e^{jkd\cos\theta}]}_{\ell_{eff}} \sin\theta \frac{e^{-jk|\mathbf{r}|}}{4\pi|\mathbf{r}|}\hat{\theta},$$

and effective length

$$\ell_{eff} = \ell[1 + e^{jkd\cos\theta}],$$

determine the gain function in terms of array directivity D.

Solution: For any linear polarized antenna we can write

$$G(\theta,\phi) = Df(\theta,\phi)$$

where function $f(\theta,\phi)$ has a maximum value of 1 and is proportional to $|\ell_{eff}\sin\theta|^2$ where θ is the angle measured from the element axis. For our two-element array described above we have

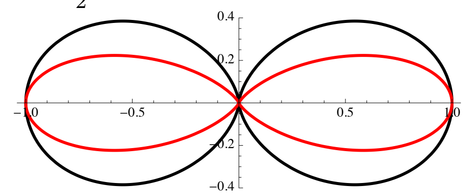
$$\begin{aligned} f(\theta,\phi) &\propto |\ell_{eff}\sin\theta|^2 \propto |1 + e^{jkd\cos\theta}|^2 \sin^2\theta \\ &= |e^{j\frac{1}{2}kd\cos\theta}(e^{j\frac{1}{2}kd\cos\theta} + e^{-j\frac{1}{2}kd\cos\theta})|^2 \sin^2\theta \\ &= |e^{j\frac{1}{2}kd\cos\theta}|^2 |e^{j\frac{1}{2}kd\cos\theta} + e^{-j\frac{1}{2}kd\cos\theta}|^2 \sin^2\theta \\ &\propto \cos^2(\frac{1}{2}kd\cos\theta)\sin^2\theta. \end{aligned}$$

The function on the right maximizes at a value of 1 when $\theta = 90^\circ$ — see its polar plot in the margin for $d = \frac{\lambda}{2}$, $d = 2\lambda$, and $d = 8\lambda$. Therefore, the gain of our two element array (for all possible d) is

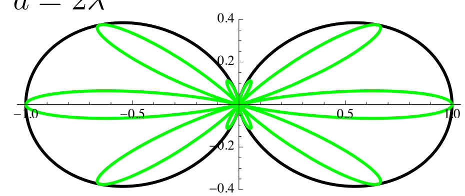
$$G(\theta,\phi) = D\cos^2(\frac{1}{2}kd\cos\theta)\sin^2\theta.$$

Polar plots of $G(\theta,\phi)/D$ for two-element array (compared to the short-dipole, shown in black):

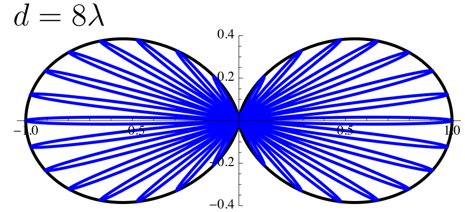
$d = \frac{\lambda}{2}$



$d = 2\lambda$



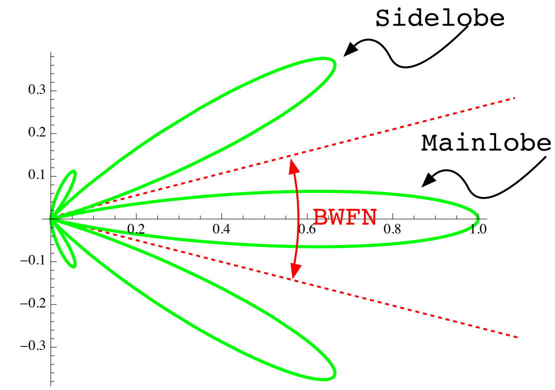
$d = 8\lambda$



Question: which of the above arrays has the largest D and smallest Ω_o?

Explain qualitatively.

$d = 2\lambda$



BWFN=
Beam-width between first nulls

Example 4: For the two-element antenna array examined in Examples 1-3, with the gain function

$$G(\theta, \phi) = D\cos^2(\frac{1}{2}kd\cos\theta)\sin^2\theta,$$

determine all angles θ for which $G(\theta, \phi) = 0$ if $d = 2\lambda$.

Solution: Clearly, $G(\theta, \phi) = 0$ at $\theta = 0^\circ$ and 180° because of $\sin^2\theta$ factor. But also, because of factor $\cos^2(\frac{1}{2}kd\cos\theta)$, we have $G(\theta, \phi) = 0$ for all θ for which

$$\frac{1}{2}kd\cos\theta = \frac{\pi}{2}(2n+1)$$

where $n = 0, \pm1, \pm2\cdots$. This condition can be satisfied when

$$\cos\theta = \frac{\pi}{kd}(2n+1) = \frac{\pi}{\frac{2\pi}{\lambda}d}(2n+1) = \frac{\lambda/2}{d}(2n+1)$$

for all integers n such that the right hand side is bounded by -1 and +1. For $d = 2\lambda$, this condition reduces to

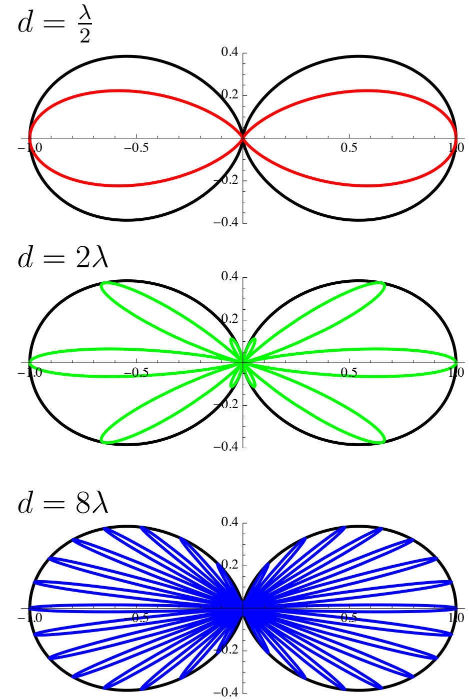
$$\cos\theta = \frac{\lambda/2}{2\lambda}(2n+1) = \frac{2n+1}{4} = \{-\frac{3}{4}, -\frac{1}{4}, \frac{1}{4}, \frac{3}{4}\}.$$

So, we have

$$G(\theta, \phi) = 0 \text{ for } \theta{=}0^\circ,\ \cos^{-1}\frac{3}{4} = 41.41^\circ,\ \cos^{-1}\frac{1}{4} = 75.52^\circ,$$

$$\cos^{-1}\frac{-1}{4} = 104.78^\circ,\ \cos^{-1}\frac{-3}{4} = 138.6^\circ,\ 180^\circ.$$

- The patterns shown for the two-element array in the margin illustrate that larger the element separation d, narrower the angular width of the mainlobe.
  - However, the number of sidelobes also increase with d, so there is no substantial directivity increase *with* distance d because of that (because of power diverted into the relatively large intensity sidelobes).
  - The remedy is to have *multiple-element arrays* analyzed next.

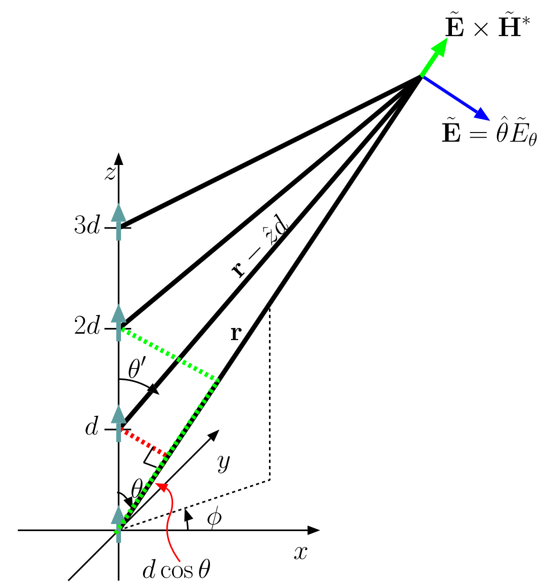


- In the two-element array the distant field, in *paraxial approximation*, was found to be

$$\tilde{\mathbf{E}}(\mathbf{r}) = \tilde{\mathbf{E}}_1(\mathbf{r}) + \tilde{\mathbf{E}}_2(\mathbf{r}) = \tilde{\mathbf{E}}_1(\mathbf{r})[1 + e^{jkd\cos\theta}]$$

after using

$$\tilde{\mathbf{E}}_2(\mathbf{r}) \approx \tilde{\mathbf{E}}_1(\mathbf{r})e^{jkd\cos\theta}.$$



- For a 3-element array with element locations $(0, 0, 0)$, $(0, 0, d)$, and $(0, 0, 2d)$ this result can be extended as

$$\tilde{\mathbf{E}}(\mathbf{r}) = \tilde{\mathbf{E}}_1(\mathbf{r}) + \tilde{\mathbf{E}}_2(\mathbf{r}) + \tilde{\mathbf{E}}_3(\mathbf{r}) = \tilde{\mathbf{E}}_1(\mathbf{r})[1 + e^{jkd\cos\theta} + e^{j2kd\cos\theta}],$$

and, for an N-element array, with elements at $(0, 0, nd)$ for n in the interval $0 \cdots N - 1$, we can write

$$\tilde{\mathbf{E}}(\mathbf{r}) = \tilde{\mathbf{E}}_1(\mathbf{r})[1 + e^{jkd\cos\theta} + e^{j2kd\cos\theta} + \cdots + e^{j(N-1)kd\cos\theta}].$$

– These superposed field expressions in the antenna far-field imply an effective length of

$$\ell_{eff} = \ell \sum_{n=0}^{N-1} (e^{jkd\cos\theta})^n.$$

- The sum on the right is called **array factor** (A.F.) and we see that the **effective length of the array antenna** is the product of the effective length ℓ of an array element *and* A.F..

- We can write the gain of the N-element array (once again as)

$$G(\theta,\phi) = Df(\theta,\phi),$$

where

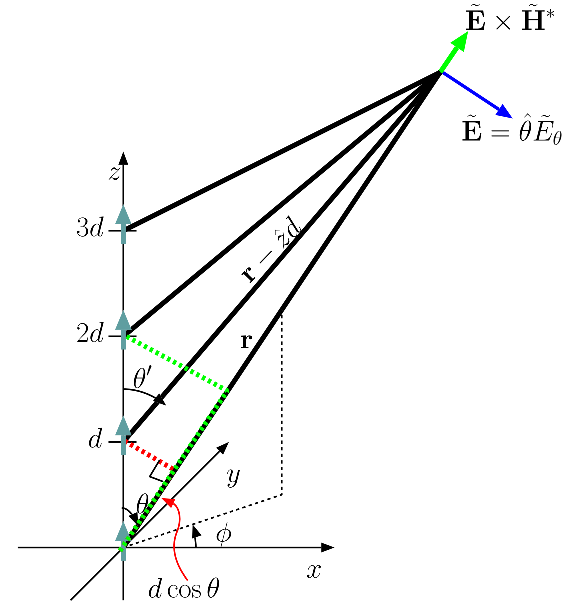
$$f(\theta,\phi) \propto |\ell_{eff}\sin\theta|^2 = |\ell|^2 |\sum_{n=0}^{N-1} (e^{jkd\cos\theta})^n|^2 \sin^2\theta$$

and has a max value of 1. The A.F. maximizes at a value of N at $\theta = 90°$ and thus it works out that

$$G(\theta,\phi) = D\sin^2\theta |\frac{1}{N}\sum_{n=0}^{N-1} (e^{jkd\cos\theta})^n|^2.$$

– To simplify this gain formula we note that

$$s \equiv 1 + w + w^2 + \cdots + w^{N-1} \quad \Rightarrow \quad sw = w + w^2 + \cdots + w^{N-1} + w^N,$$



The same interference principle governs N-element arrays: at locations where field phasors from individual elements have the same angle, constructive interference takes place, and the radiation field of the array is strong. At other locations where field phasors from individual elements cancel one another, the field of the whole array is weak.

and, therefore,

$$s(w-1) = w^N - 1 \quad \Rightarrow \quad s = \frac{w^N - 1}{w - 1}.$$

- Applying this summation formula for s with $w = e^{jkd\cos\theta}$, we obtain

$$\text{A.F.} = \sum_{n=0}^{N-1} (e^{jkd\cos\theta})^n = \frac{e^{jNkd\cos\theta} - 1}{e^{jkd\cos\theta} - 1}.$$

Now,

$$\begin{aligned} |\text{A.F.}| &= |\sum_{n=0}^{N-1} (e^{jkd\cos\theta})^n| = \frac{|e^{jNkd\cos\theta} - 1|}{|e^{jkd\cos\theta} - 1|} \\ &= \frac{|e^{j\frac{N}{2}kd\cos\theta}(e^{j\frac{N}{2}kd\cos\theta} - e^{-j\frac{N}{2}kd\cos\theta})|}{|e^{j\frac{1}{2}kd\cos\theta}(e^{j\frac{1}{2}kd\cos\theta} - e^{-j\frac{1}{2}kd\cos\theta})|} = \frac{|\sin(\frac{N}{2}kd\cos\theta)|}{|\sin(\frac{1}{2}kd\cos\theta)|}. \end{aligned}$$

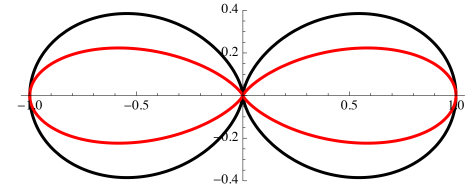
The upshot is,

$$G(\theta, \phi) = D\sin^2\theta \frac{\sin^2(\frac{N}{2}kd\cos\theta)}{N^2\sin^2(\frac{1}{2}kd\cos\theta)}$$

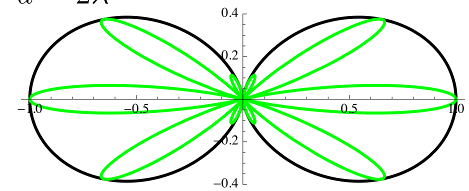
for an N-element array with a physical size Nd.

  – Plots of $G(\theta, \phi)/D$ for $d = \frac{\lambda}{2}$ and $N = 2$, 4, 16 are shown in the margin. Note the reduced sidelobe levels (you can barely see them) and the fact that larger N implies larger directivity D.

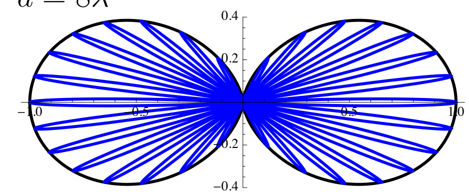
$d = \frac{\lambda}{2}$



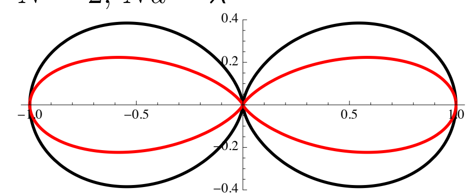
$d = 2\lambda$



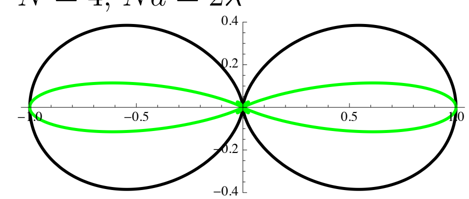
$d = 8\lambda$



$N = 2,\ Nd = \lambda$



$N = 4,\ Nd = 2\lambda$



$N = 16,\ Nd = 8\lambda$