10 Antenna gain, beam pattern, directivity

- A dipole antenna (or a closely related monopole to be studied in Lecture 18) is a "near perfect" radiator for purposes of "broadcasting" — that is, sending waves of equal amplitudes in all directions to reach out multiple targets or receivers.

- However dipole is a poor choice when the objective is to radiate the power P_{rad} in a specific direction (i.e., towards a specific receiver), as in

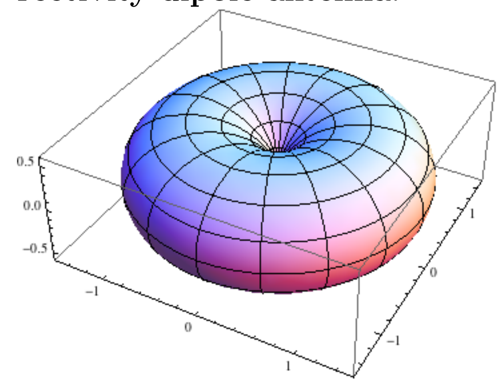
  - **communication** with deep space probes or orbiting satellites, or with
  - **radar beacons** where the objective is to determine the direction of a moving target.

  In such applications we need **high-gain** and **directive** antennas, as opposed to low-gain and non-directive antennas such as a single dipole.

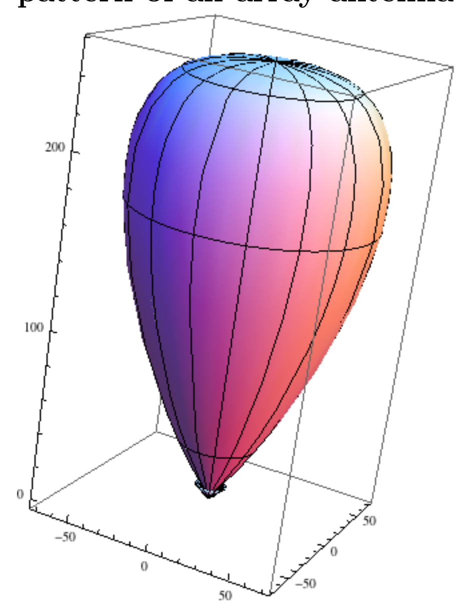
- Qualitatively speaking, gain and directivity of an antenna measures its ability to confine its radiated wave fields within a narrow field of view called the **antenna beam** or **beam pattern**.

  - when a narrow antenna beam is achieved, and all the radiated power P_{rad} of the antenna is conveyed through this beam, the power density of the waves is naturally high within the beam.

Beam pattern plot of low directivity dipole antenna:

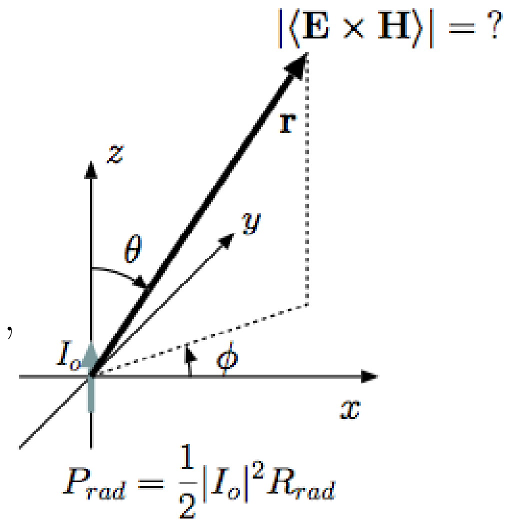


A higher directivity beam pattern of an array antenna



Arrays **of dipoles can serve as high-gain antennas needed in beaming applications as we will learn in the next lecture.**

In this lecture we will focus on the definition of antenna gain and directivity as well as the related concept of beam solid angle.



Power density of a radiating antenna in the far field

- Consider an antenna located at the origin with an input current of I_o, radiation resistance R_{rad}, and a radiated power

$$P_{rad} = \frac{1}{2}|I_o|^2 R_{rad}\,\text{Watts}.$$

What would be the time-average Poynting magnitude $|\langle \mathbf{E}(\mathbf{r},t) \times \mathbf{H}(\mathbf{r},t)\rangle|$, i.e., the power density in Watts/m^2 of the radiation fields at a location $\mathbf{r} = (r, \theta, \phi)$ a distance r away from the antenna?

Answer:

– If the antenna were an **isotropic radiator** then we would have a **power density** of

$$|\langle \mathbf{E}(\mathbf{r},t) \times \mathbf{H}(\mathbf{r},t)\rangle| = \frac{P_{rad}}{4\pi r^2};$$

however, no real antenna is an isotropic radiator, and thus the correct answer can be formally cast as

$$|\langle \mathbf{E}(\mathbf{r},t) \times \mathbf{H}(\mathbf{r},t)\rangle| = \frac{P_{rad}}{4\pi r^2} G(\theta, \phi)$$

in terms of an **antenna gain** (over isotropic radiator)

$$G(\theta,\phi) \equiv \frac{|\langle \mathbf{E}(\mathbf{r},t) \times \mathbf{H}(\mathbf{r},t)\rangle|}{\frac{P_{rad}}{4\pi r^2}},$$

to be determined.

Clearly, gain $G(\theta,\phi)$ is the ratio of the radiated average power density of an antenna to that of an **isotropic radiator** (hypothetical perfect broadcasting antenna) radiating the same average power P_{rad}.

– According to this definition, the solid angle integral of gain $G(\theta,\phi)$ is

$$\int d\Omega G(\theta,\phi) \equiv \frac{4\pi \int d\Omega r^2 |\langle \mathbf{E}(\mathbf{r},t) \times \mathbf{H}(\mathbf{r},t)\rangle|}{P_{rad}} = \frac{4\pi \oint \langle \mathbf{E}(\mathbf{r},t) \times \mathbf{H}(\mathbf{r},t)\rangle \cdot d\mathbf{S}}{P_{rad}} = 4\pi, \text{ a fixed value.}$$

Since

$$G(\theta,\phi) \propto |\langle \mathbf{E} \times \mathbf{H}\rangle| \propto |\ell \sin\theta|^2,$$

we can write

$$G(\theta,\phi) = K|\ell \sin\theta|^2$$

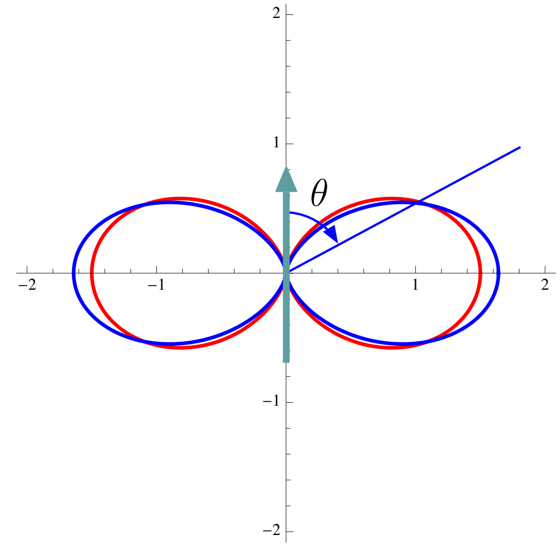
in terms of a proportionality constant K, which is subsequently identified as

$$K = \frac{4\pi}{\int d\Omega |\ell \sin\theta|^2} \text{ after applying the constraint } \int d\Omega G(\theta,\phi) = 4\pi.$$

Thus we obtain a general gain formula

$$G(\theta, \phi) = \frac{4\pi|\ell \sin\theta|^2}{\int d\Omega |\ell \sin\theta|^2}$$

applicable to all antennas for which the foreshortened effective length $\ell \sin\theta$ is known.



Gain functions $G(\theta, \phi)$ depicted on a constant ϕ plane for
(a) short-dipole (red curve), and

(b) half-wave dipole (blue curve).

- For an arbitrary antenna, gain calculation can be complicated because of the solid angle integral in the denominator in $G(\theta, \phi)$ formula.

  However, for a short dipole with $\ell = L/2$ the calculation is simple and leads to (in case of $\hat{z}$-polarization)

$$G(\theta, \phi) = \frac{4\pi|\sin\theta|^2}{\int d\Omega |\sin\theta|^2} = \frac{4\pi|\sin\theta|^2}{2\pi\frac{4}{3}} = \frac{3}{2}\sin^2\theta.$$

  For a half-wave dipole it works out that

$$G(\theta, \phi) \approx 1.64 \frac{\cos^2(\frac{\pi}{2}\cos\theta)}{\sin^2\theta}$$

  with a maximum value of 1.64 at $\theta = 90°$.

  – Having maximum gains of 1.5 and 1.64, respectively, short- and half-wave-dipoles are considered to be **low-directivity** antennas.

Directivity D of any antenna is defined to be the maximum value of its gain $G(\theta,\phi)$, i.e.,

$$D = G(\theta,\phi)_{max}.$$

While the solid angle integral of $G(\theta,\phi)$ is constrained to have a fixed value of 4π, there is no constraint on the maximum value of $G(\theta,\phi)$; therefore, it is possible to design antennas with arbitrarily large directivities D by making the antenna beam shape arbitrarily narrow.

- Note that the constraint

$$\int d\Omega G(\theta,\phi) = 4\pi$$

implies

$$D\int d\Omega \frac{G(\theta,\phi)}{G(\theta,\phi)_{max}} = 4\pi,$$

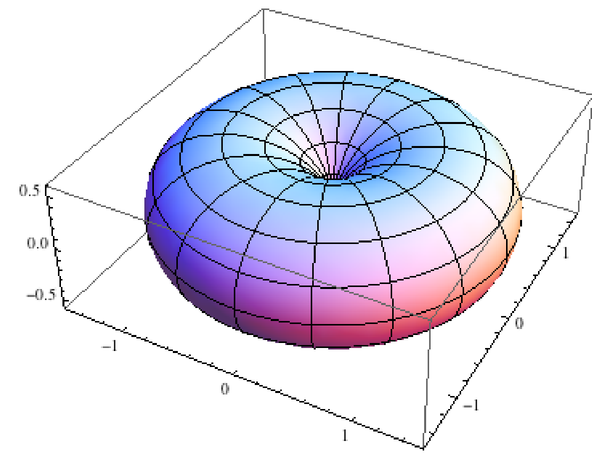
which can also be written as

$$D\Omega_o = 4\pi.$$

in terms of **beam solid angle**

$$\Omega_o \equiv \int d\Omega \frac{G(\theta,\phi)}{G(\theta,\phi)_{max}}$$

to be discussed further in this lecture.



Gain function

$$G(\theta,\phi) = \frac{3}{2}\sin^2\theta$$

of *short-dipole* depicted as a 3D polar plot — gain in any direction (θ,ϕ) is proportional to the radius vector from the origin to the depicted surface.

A short-dipole has a low directivity of

$$D = 1.5$$

because it radiates with a broad beam that is isotropic in azimuth.

Antennas with high-directivity have narrow and pointy beam shapes.

- **Important result:** the product of antenna directivity D and the beam solid angle Ω_o is fixed, specifically

$$D\Omega_o = 4\pi,$$

which implies that if D is large then Ω_o is small and vice versa.

Antennas with high-directivity have narrow and pointy beam shapes.

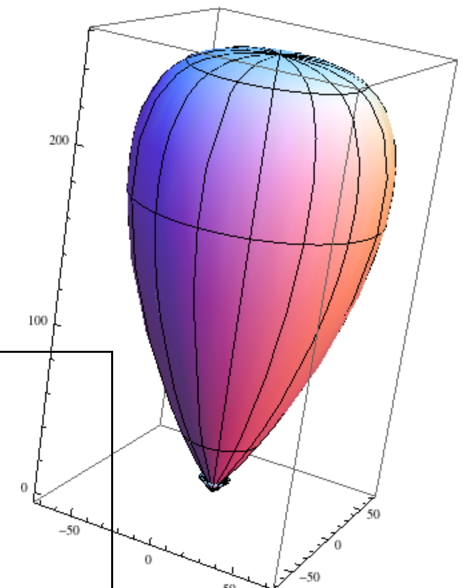
- A useful method to determine the antenna directivity is to use

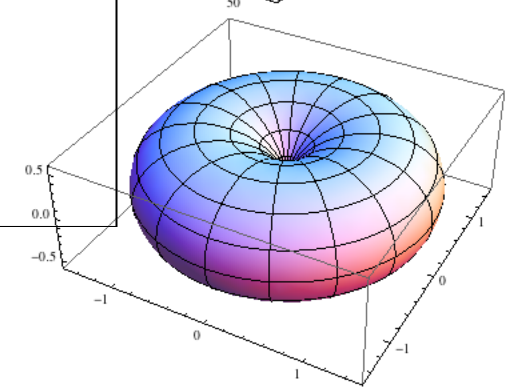
$$D = \frac{4\pi}{\Omega_o},$$

where the solid angle

$$\Omega_o = \int d\Omega \frac{G(\theta,\phi)}{G(\theta,\phi)_{max}} = \int d\Omega \frac{|\ell \sin\theta|^2}{|\ell \sin\theta|^2_{max}}$$

can be calculated once the antenna effective length is known.





Example 1: For a short dipole with $\ell = L/2$, we have

$$\Omega_o = \int d\Omega \frac{|\sin\theta|^2}{|\sin\theta|^2_{max}} = \int d\Omega |\sin\theta|^2 = 2\pi\frac{4}{3} = \frac{8\pi}{3}.$$

Consequently,

$$D = \frac{4\pi}{\Omega_o} = \frac{4\pi}{8\pi/3} = 1.5$$

consistent with what we learned above.

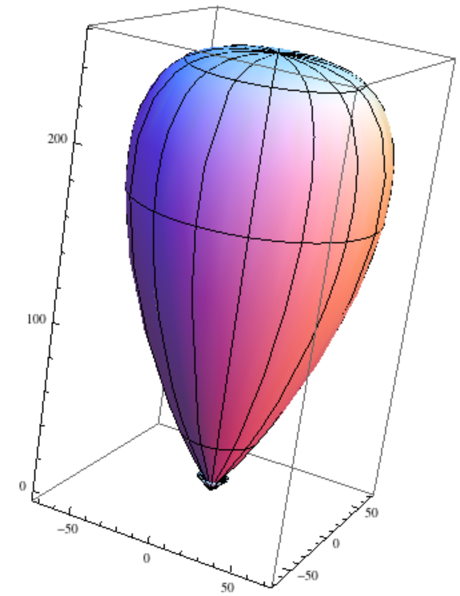
  - This method of finding D from Ω_o is very useful because there are geometrical methods for estimating Ω_o in terms of the physical antenna size (as we will learn later on).

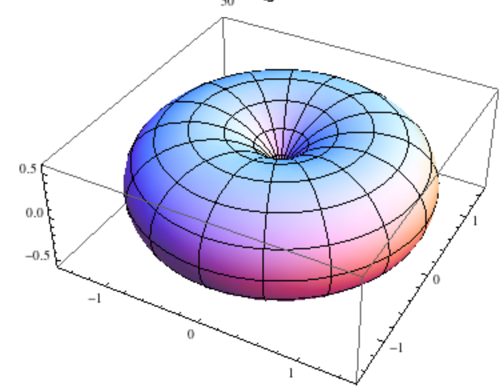
Once D is determined, the gain of the antenna can be written as

$$G(\theta, \phi) = D \frac{|\ell \sin\theta|^2}{|\ell \sin\theta|^2_{max}}$$

without the need to perform a solid angle integral in practice.

Antennas with high-directivity have narrow and pointy beam shapes.





- The beam solid angle

$$\Omega_o = \int d\Omega \frac{G(\theta, \phi)}{G(\theta, \phi)_{max}} = \int d\Omega \frac{|\ell \sin\theta|^2}{|\ell \sin\theta|^2_{max}}$$

  extends the concept of "angle" from 2D to 3D to describe the angular width of the antenna beam pattern. Let us examine this parameter more closely.

- Ordinary **angles** ranging from 0 to 2π **radians** (with degree equivalents ranging from 0 to 360) correspond to **arc lengths** measured on unit-radius circles drawn on 2D planar surfaces.

- **Solid angles** ranging from 0 to 4π **steradians** correspond to **areas of patches** or **spots** specified on unit-radius spheres defined in 3D space.

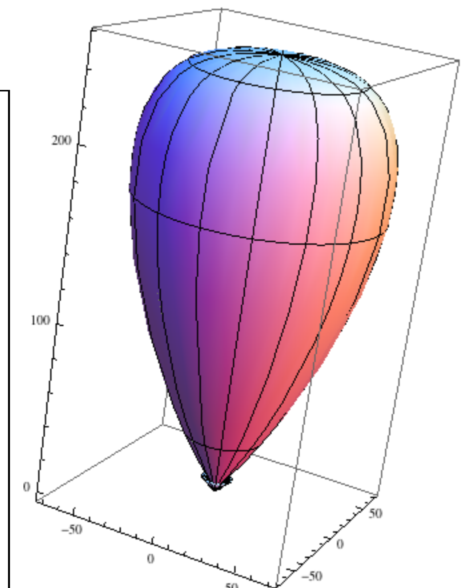
  – An **antenna-beam solid angle**

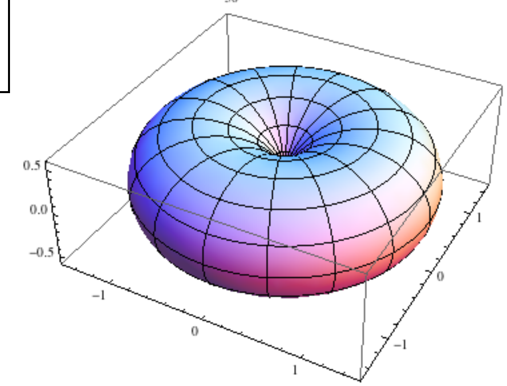
$$\Omega_o = \int d\Omega \frac{G(\theta, \phi)}{G(\theta, \phi)_{max}} = \int d\Omega \frac{|\langle \mathbf{E} \times \mathbf{H} \rangle|}{|\langle \mathbf{E} \times \mathbf{H} \rangle|_{max}}$$

  is an **equivalent area** of a spot or a patch (centered about the direction of $|\langle \mathbf{E} \times \mathbf{H} \rangle|_{max}$) specified on a unit sphere surrounding the antenna, having the property that the entire power output P_{rad} of the antenna would flood this area with an equal flux density of $|\langle \mathbf{E} \times \mathbf{H} \rangle|_{max}$ if the beam were reformed into a conical shape.

– Beam shapes of high-directivity antennas with small Ω_o can be well represented by equivalent conical beams, but such a representation is not appropriate to dipole-like broadcast antennas (see margin).

Antennas with high-directivity have narrow and pointy beam shapes.





Example 2: For a short dipole with

$$G(\theta, \phi) = \frac{3}{2} \sin^2 \theta$$

we have

$$D = \frac{3}{2} \quad \text{and} \quad \Omega_o = \int d\Omega \frac{|\sin\theta|^2}{|\sin\theta|^2_{max}} = 2\pi \frac{4}{3} = \frac{8\pi}{3}$$

as we already established in Example 1.

Consequently,

$$D = \frac{4\pi}{\Omega_o} = \frac{4\pi}{8\pi/3} = 1.5$$

consistent with what we learned above.

Example 3: An antenna designer comes up with a model that has a gain function specified as

$$G(\theta, \phi) = \begin{cases} D\sin^2\theta, & 0 < \theta < \frac{\pi}{2} \\ 0, & \text{othewise,} \end{cases}$$

where D is the antenna directivity. Determine both D and the beam solid angle Ω_o.

Solution: Since the solid angle integral of $G(\theta, \phi)$ has to equal 4π, it must be true that

$$\int d\Omega G(\theta, \phi) = \int_{\phi=0}^{2\pi} d\phi \int_{\theta=0}^{\pi/2} d\theta \sin\theta D \sin^2\theta = 4\pi.$$

It follows that

$$2\pi D \int_{\theta=0}^{\pi/2} d\theta \sin\theta \sin^2\theta = 4\pi \quad \Rightarrow \quad -D \int_{\theta=0}^{\pi/2} (d\cos\theta)(1 - \cos^2\theta) = 2$$

from which we get

$$D = \frac{2}{\int_{\theta=\pi/2}^{0} d\cos\theta(1-\cos^2\theta)} = \frac{2}{(\cos\theta - \frac{\cos^3\theta}{3}|_{\pi/2}^{0}} = \frac{2}{1-\frac{1}{3}} = 3.$$

This is twice the directivity of the short-dipole (which makes sense because half the gain function of the short dipole is missing from the gain of this antenna).

As for the beam solid angle, it is

$$\Omega_o = \frac{4\pi}{D} = \frac{4\pi}{3},$$

which is half the solid angle of a short dipole (again for the same reason).