9 Poynting vector, radiated power, radiation resistance

Consider the radiation fields of a $\hat{z}$-polarized short-dipole antenna shown in the margin in a compact form.

- How much average power is radiated by the short-dipole antenna to sustain these fields, and

- how can we determine this amount, P_{rad}, by electrical measurements which can be performed at the antenna input port — the small gap at the dipole center where the dipole is connected to the source circuit (typically via some transmission line network)?

To answer these questions we will calculate in this lecture the **average Poynting vector** of radiation fields of the dipole antenna and the "flux" of the same vector computed over a sphere imagined to surround the dipole.

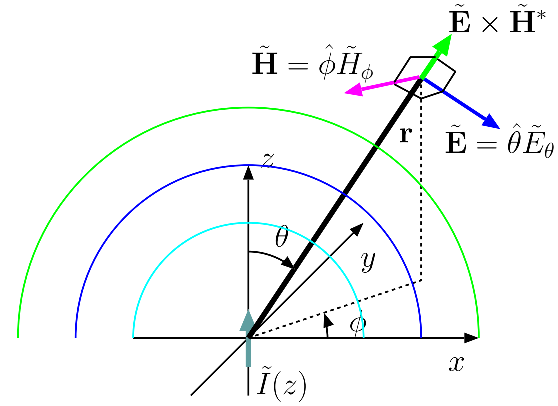
- Recall once again that Poynting vector

$$\mathbf{S} \equiv \mathbf{E} \times \mathbf{H}$$

denotes the energy transported by electromagnetic fields per unit time and per unit area normal to the vector itself. With time-harmonic fields the average value of Poynting vector can be denoted and computed as

$$\langle \mathbf{E} \times \mathbf{H} \rangle = \frac{1}{2}\text{Re}\{\tilde{\mathbf{E}} \times \tilde{\mathbf{H}}^*\}$$

in terms of field phasors $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{H}}$.



Radiation fields of short dipole:

$$\tilde{\mathbf{E}} = \tilde{E}_\theta \hat{\theta}$$

and

$$\tilde{\mathbf{H}} = \frac{\tilde{E}_\theta}{\eta_o}\hat{\phi}$$

where

$$\tilde{E}_\theta = j\eta_o I_o k\ell \sin\theta \frac{e^{-jkr}}{4\pi r}$$

and $\ell = L/2$**.**

- It is this quantity

$$\langle \mathbf{S} \rangle = \langle \mathbf{E} \times \mathbf{H} \rangle$$

which is independent of the storage fields of dipole antennas and only depend on their radiation fields.

- Using (see margin once again)

$$\tilde{\mathbf{E}} = \hat{\theta}\tilde{E}_\theta \quad \text{and} \quad \tilde{\mathbf{H}} = \hat{\phi}\frac{\tilde{E}_\theta}{\eta_o} \quad \text{with} \quad \tilde{E}_\theta = j\eta_o I k\ell \sin\theta \frac{e^{-jkr}}{4\pi r},$$

we find

$$\tilde{\mathbf{E}} \times \tilde{\mathbf{H}}^* = \hat{\theta}\tilde{E}_\theta \times (\hat{\phi}\frac{\tilde{E}_\theta}{\eta_o})^* = \hat{\theta} \times \hat{\phi}\frac{|\tilde{E}_\theta|^2}{\eta_o} = \hat{r}\frac{|\tilde{\mathbf{E}}|^2}{\eta_o}$$

and

$$\langle \mathbf{E} \times \mathbf{H} \rangle = \frac{1}{2}\mathrm{Re}\{\tilde{\mathbf{E}} \times \tilde{\mathbf{H}}^*\} = \hat{r}\frac{|\tilde{\mathbf{E}}|^2}{2\eta_o}.$$

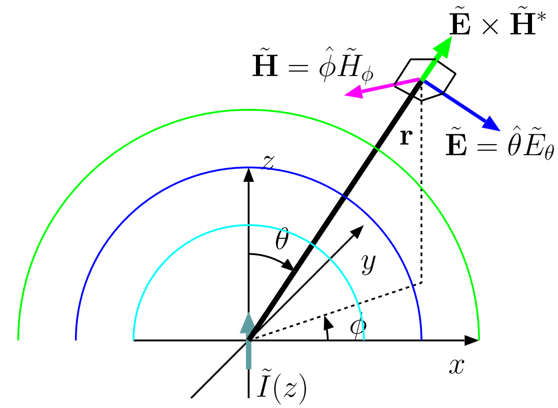
- Since

$$|\tilde{\mathbf{E}}|^2 = |\tilde{E}_\theta|^2 = \frac{\eta_o^2 |I_o|^2 k^2 |\ell|^2 \sin^2\theta}{(4\pi r)^2} = \frac{\eta_o^2 |I_o|^2 |\ell|^2 \sin^2\theta}{4(\lambda r)^2},$$

we have

$$\langle \mathbf{E} \times \mathbf{H} \rangle = \hat{r}\frac{|\tilde{\mathbf{E}}|^2}{2\eta_o} = \frac{\eta_o}{8}|I_o|^2 \frac{|\ell|^2}{(\lambda r)^2} \sin^2\theta \hat{r}.$$

- The expression above is the **energy flux density** or **transmitted power density** of the dipole antenna as a function of distance r from the dipole and angle θ of viewing direction off the dipole axis.



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and $\ell = L/2$**.**

- The average power output of the dipole — radiated power P_{rad} — can next be obtained by computing the flux of $\langle \mathbf{E} \times \mathbf{H} \rangle$ over any closed surface surrounding the dipole.

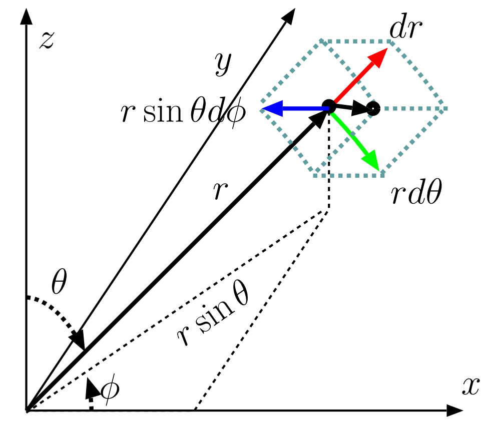
  – This calculation is most easily carried out over a spherical surface of radius r having infinitesimal surface elements

$$d\mathbf{S} = \hat{r}(r \sin\theta d\phi)(r d\theta) = \hat{r} r^2 \sin\theta d\theta d\phi \equiv \hat{r} r^2 d\Omega,$$

  where

$$d\Omega \equiv \sin\theta d\theta d\phi$$

  (introduced to maintain a compact notation) is called a **solid angle** increment.



Infinitesimal area on a constant r surface is

$$dS = (rd\theta)(r\sin\theta d\phi) = r^2 d\Omega$$

where

$$d\Omega \equiv \sin\theta d\theta d\phi$$

is called infinitesimal solid angle.

We then note that

$$\langle \mathbf{E} \times \mathbf{H} \rangle \cdot d\mathbf{S} = \frac{\eta_o}{8\lambda^2} |I_o|^2 |\ell|^2 \sin^2\theta d\Omega$$

and the flux of $\langle \mathbf{E} \times \mathbf{H} \rangle$ is

$$\underbrace{\oint \langle \mathbf{E} \times \mathbf{H} \rangle \cdot d\mathbf{S}}_{P_{rad}} = \frac{\eta_o}{8\lambda^2} |I_o|^2 \int d\Omega |\ell|^2 \sin^2\theta$$

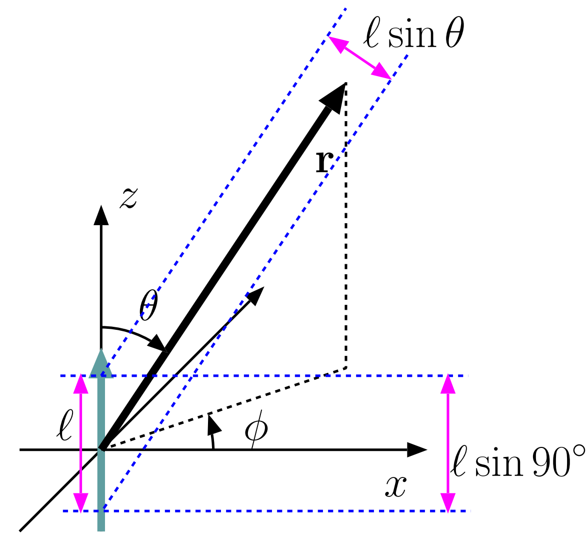
where it is implied that

$$\int d\Omega = \int_{\phi=0}^{2\pi} d\phi \int_{\theta=0}^{\pi} d\theta \sin\theta.$$

– This result can be cast as

$$P_{rad} = \frac{1}{2} R_{rad} |I_o|^2 \quad \text{where} \quad R_{rad} = \frac{\eta_o}{4\lambda^2} \int d\Omega |\ell \sin\theta|^2$$

is known as **radiation resistance**.



Note: recall that ℓ may be a function of θ itself!

- If ℓ is the *effective length* of a dipole — distinct from its physical length L because of current weighting — then $\ell \sin\theta$ is "how long the effective length looks" when one sees it (the dipole) at an angle (see margin).

  Solid angle integral of the square of this "foreshortened" effective length, namely

  $$\int d\Omega |\ell \sin\theta|^2,$$

  determines the radiation resistance of the dipole antenna.

  Since for a short dipole $\ell = \frac{L}{2}$ is independent of angle θ (unlike for half-wave dipole), we have

$$\begin{aligned} \int d\Omega |\ell \sin\theta|^2 &= (\frac{L}{2})^2 \int d\Omega |\sin\theta|^2 = (\frac{L}{2})^2 \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta |\sin\theta|^2 \\ &= (\frac{L}{2})^2 2\pi \underbrace{\int_0^{\pi} d\theta \sin\theta |\sin\theta|^2}_{4/3} = \frac{2\pi L^2}{3}. \end{aligned}$$

  Hence the radiation resistance of the short dipole is

$$R_{rad} = \frac{\eta_o}{4\lambda^2} \int d\Omega |\ell \sin\theta|^2 = \frac{\eta_o}{4\lambda^2} \frac{2\pi L^2}{3} = 20\pi^2 (\frac{L}{\lambda})^2 \, \Omega.$$

- Since a short dipole is constrained to have $\frac{L}{\lambda} \ll 1$, say $\frac{1}{10}$ or smaller, R_{rad} will be equal to or less than about $2\,\Omega$.

- Thus, a short dipole with an input current of $\tilde{I}(0) = I_o = 1$ A will have at best an average power output $\frac{1}{2} I_o^2 R_{rad}$ of about 1 W.

  This is not quite at the level of 100s of W's of power that typical radio stations transmit!

  Using antennas with higher R_{rad} than a short dipole[1] — e.g., a half-wave dipole for which $R_{rad} \approx 73\,\Omega$ — is the best way of addressing this difficulty since the alternate solution of increasing I_o (as needed) is not recommended because of *antenna losses*:

  – In practice, antennas appear as a circuit element with input resistance

  $$R_o = R_{rad} + R_{loss}$$

  where R_{loss} represents ohmic losses (heating of antenna wires) — an antenna consumes an average power of

  $$\frac{1}{2} I_o^2 (R_{rad} + R_{loss})$$

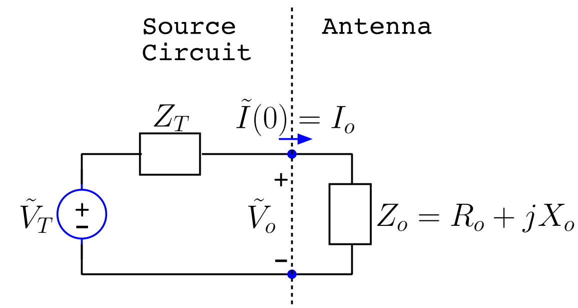
  out of which only

  $$\frac{1}{2} I_o^2 R_{rad}$$

[1]Short dipoles are typically employed as receiving antennas rather than transmitting antennas because of this. Receiving properties of antennas are closely related to their transmission properties, but figures of merit of antennas pertinent in transmission and reception are somewhat different as we will learn later on in the course.

is the useful radiated power.

Typically $R_{loss} \propto L$, whereas $R_{rad} \propto L^2$ for small L, so going to longer dipoles (and learning more about them in ECE 454) really helps.



- A source circuit connected to the antenna terminals "sees" the antenna (and the radiation volume with which the antenna interacts) as a two-terminal element having some impedance

$$Z_o \equiv \frac{\tilde{V}_o}{\tilde{I}(0)} = R_o + jX_o$$

known as **antenna impedance**.

  - We have already discussed the **resistive component** R_o above.
  - Modeling the **reactive component** X_o requires working with storage fields of the antenna as well, matching components of total fields to proper boundary conditions imposed by the actual surfaces of antenna wires (i.e., antenna geometry needs to be specified in detail before X_o can be determined).

    Antenna reactance will be examined in some detail in ECE 454 (along with methods of calculating $\tilde{I}(z)$).
  - We will not need to calculate antenna reactances in this course. However, it is worth mentioning that
    1. antennas with $X_o = 0$ are known as **resonant antennas**, and
    2. half-wave dipole is a resonant antenna (see margin note).

Antenna reactance:

Short dipoles **have** ***capacitive reactances*****, just like the line-impedance at a small distance away from an open termination on a transmission line.**

Capacitive reactance switches to an inductive one when the dipole length is about $\lambda/2$, just like the line-impedance at a distance $\lambda/4$ away from an open termination.

Thus the ***half-wave dipole*** **is** ***resonant*****, having a zero input reactance — $Z_o = R_{rad} + j0$ for an ideal half-wave dipole.**

In practice, resonant half-wave dipole with a length L and wire radius a has $L + 2a = \lambda/2$ to a good approximation.