6 Spherical waves

In this lecture we will find out that short-filaments of oscillatory currents produce *uniform* **spherical waves** of vector potential propagating away from the filament. The relationship between spherical waves of vector potential and the corresponding electromagnetic wave fields will be examined in the next lecture.

We recall that time-varying solutions of Maxwell's equations can be obtained via

$$\mathbf{B} = \nabla \times \mathbf{A},$$

where the vector potential $\mathbf{A}(\mathbf{r}, t)$ is related to time-varying current density $\mathbf{J}(\mathbf{r}, t)$ via

Time-domain:

$$\mathbf{A}(\mathbf{r}, t) = \int \frac{\mu_o \mathbf{J}(\mathbf{r}', t - \frac{|\mathbf{r}-\mathbf{r}'|}{c})}{4\pi|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}'.$$

Frequency-domain:

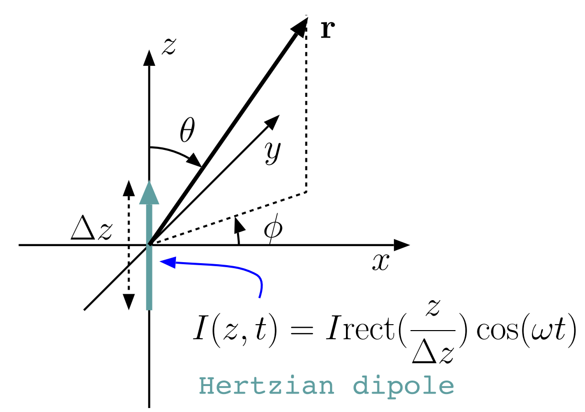
$$\tilde{\mathbf{A}}(\mathbf{r}) = \int \frac{\mu_o \tilde{\mathbf{J}}(\mathbf{r}') e^{-jk|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}',$$

where

$$k = \omega\sqrt{\mu_o \epsilon_o}.$$

- We will next examine the implications of the above results from Lecture 4 for an $\hat{z}$ directed **infinitesimal current filament** defined as

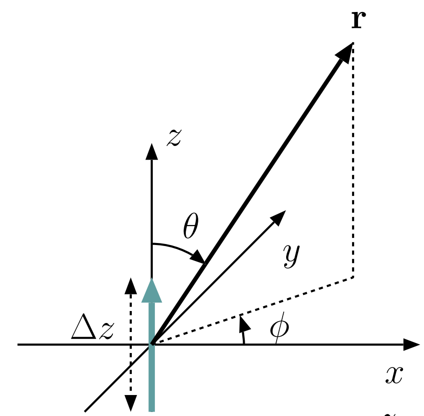
$$I(\mathbf{r}, t) = \begin{cases} I\cos(\omega t), & \text{for } x = 0,\ y = 0,\ -\frac{\Delta z}{2} < z < \frac{\Delta z}{2} \\ 0, & \text{otherwise.} \end{cases}$$



where constant I is specified in units of amperes (A). We can associate with this infinitesimal current the following current density function

$$\mathbf{J}(\mathbf{r},t) = \begin{cases} I\delta(x)\delta(y)\cos(\omega t)\hat{z}, & \text{for } -\frac{\Delta z}{2} < z < \frac{\Delta z}{2} \\ 0, & \text{otherwise.} \end{cases}$$
$$= I\delta(x)\delta(y)\text{rect}(\frac{z}{\Delta z})\cos(\omega t)\hat{z}\,\frac{\text{A}}{\text{m}^2}$$

recalling that the dimension of an impulse $\delta(x)$ is m^{-1}.



$\mathbf{J}(\mathbf{r},t) = I\delta(x)\delta(y)\text{rect}(\frac{z}{\Delta z})\cos(\omega t)\hat{z}$

- The oscillatory and $\hat{z}$ directed infinitesimal current filament of a length Δz can in turn can be represented in terms of a *phasor*

$$\tilde{\mathbf{J}}(\mathbf{r}) = I\delta(x)\delta(y)\text{rect}(\frac{z}{\Delta z})\hat{z}\,\frac{\text{A}}{\text{m}^2}.$$

We can also re-write this as

$$\tilde{\mathbf{J}}(\mathbf{r}) = I\Delta z\,\delta(x)\delta(y)\,\frac{\text{rect}(\frac{z}{\Delta z})}{\Delta z}\hat{z}\,\frac{\text{A}}{\text{m}^2}$$

in which the ratio with the rectangle in the numerator can be treated as "$\delta(z)$" provided that

  - the width, Δz, of the rectangle is considered an *infinitesimal* so that the ratio

$$\frac{\text{rect}(\frac{z}{\Delta z})}{\Delta z}$$

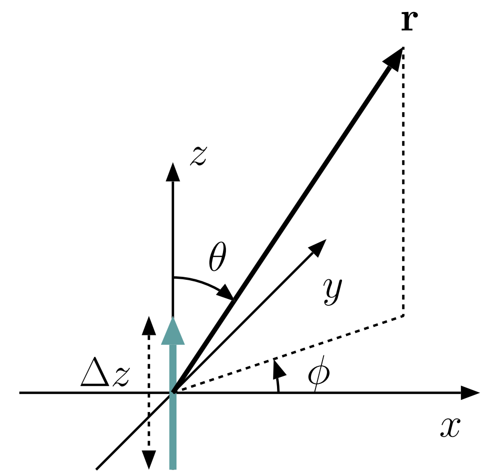
represents *in effect* an infinitely thin and tall function centered about $z = 0$ having a unity area underneath.

- Next we substitute this current density phasor $\tilde{\mathbf{J}}(\mathbf{r})$ (with Δz considered an infinitesimal) into the phasor formula for the retarded vector potential to obtain

$$\tilde{\mathbf{A}}(\mathbf{r}) = \int \frac{\mu_o \tilde{\mathbf{J}}(\mathbf{r}')e^{-jk|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} d^3\mathbf{r}'$$

$$= \int\int\int \frac{\mu_o \overbrace{I\Delta z \delta(x')\delta(y')\delta(z')\hat{z}}^{\tilde{\mathbf{J}}(\mathbf{r}')} e^{-jk|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} dx'dy'dz'$$

where the integrations are to be carried over x', y', and z' in the range $-\infty$ to $+\infty$.



$\tilde{\mathbf{J}}(\mathbf{r}) = I\Delta z \delta(x)\delta(y)\delta(z)\hat{z}$

- These are very easy integrals to take because of $\delta(x')$, $\delta(y')$, and $\delta(z')$ factors in the integrand, and lead to (after replacing all x', y', and z' elsewhere in the integrand by 0)

$$\tilde{\mathbf{A}}(\mathbf{r}) = \frac{\mu_o}{4\pi} I\Delta z \frac{e^{-jkr}}{r}\hat{z},$$

where $r = |\mathbf{r}|$ as usual. Converting this result into time domain by multiplying it with $e^{j\omega t}$ and taking the real part of the product we obtain

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_o}{4\pi} I\Delta z \frac{\cos(\omega t - kr)}{r}\hat{z}.$$

We have just finished deriving the retarded vector potential solution of an oscillatory infinitesimal current filament known as the **Hertzian dipole**.

Our results indicate that for a Hertzian dipole oriented in $\hat{z}$ direction, the vector potential solution

Frequency-domain:

$$\tilde{\mathbf{A}}(\mathbf{r}) = \frac{\mu_o}{4\pi} I \Delta z \frac{e^{-jkr}}{r} \hat{z}$$

Time-domain:

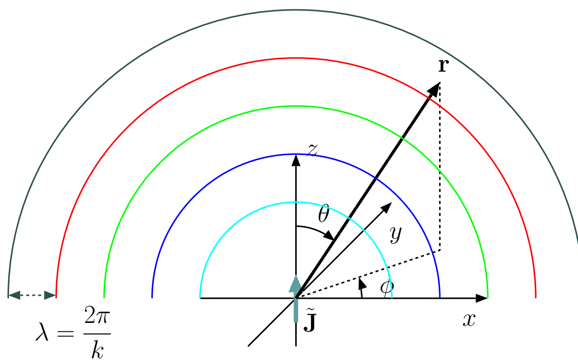
$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_o}{4\pi} I \Delta z \frac{\cos(\omega t - kr)}{r} \hat{z}$$

is also oriented in the $\hat{z}$ direction and oscillate in time at the frequency ω of the oscillating dipole. Note that:

1. These vector potential solutions describe a **spherical wave** (as opposed to a plane wave) characterized by **spherical surfaces of constant phase** associated with

$$e^{-jkr} \quad \text{and} \quad \cos(\omega t - kr)$$

   variations in frequency and time domains.



2. Spherical wave solution is *uniform* in the sense that the vector potential phasor $\tilde{\mathbf{A}}$ is constant (in direction and magnitude) on spherical surfaces of constant phase (in analogy to uniform TEM plane waves of electric and magnetic fields studies in ECE 329).

3. Clearly, the propagation speed of the spherical wave is

$$v_p = \frac{\omega}{k} = \frac{\omega}{\omega\sqrt{\mu_o \epsilon_o}} = c.$$

4. The spherical wave is also characterized by an oscillation amplitude that varies as $\frac{1}{r}$ away from the radiating source

- In the next lecture we will take the curl of this result (using spherical coordinates operators) to obtain spherical (but non-uniform) waves of $\mathbf{B}$ that accompany the $\mathbf{A}$-waves, and then derive the accompanying spherical (but non-uniform) $\mathbf{E}$-waves using Ampere's law.
  - We will find out $\mathbf{E}$- and $\mathbf{B}$-waves derived from $\mathbf{A}$-waves are in general non-uniform and form "beams" of directions along which field magnitudes $|\tilde{\mathbf{E}}|$ and $|\tilde{\mathbf{B}}|$ maximize over spherical planes of constant phase.
  - The mathematical description of these *beams* is provided by the "gain function" and the "solid angle" of the radiating system to be defined and explored in Lecture 10.

- In deriving $\mathbf{E}$- and $\mathbf{B}$-waves from $\mathbf{A}$ we will not explicitly worry about $V(\mathbf{r}, t)$ and $\rho(\mathbf{r}, t)$ that accompanies the Hertzian dipole behavior (since $\mathbf{J}$ contains all information included in ρ variations).

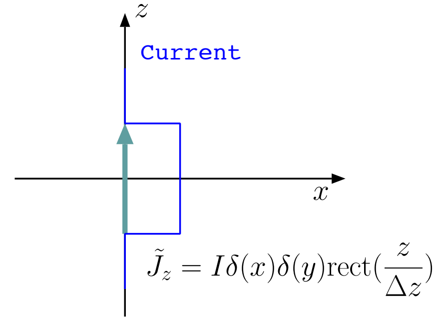
- For completeness sake, however, let us examine what kind of $\rho(\mathbf{r}, t)$ variation should be expected for the Hertzian dipole.

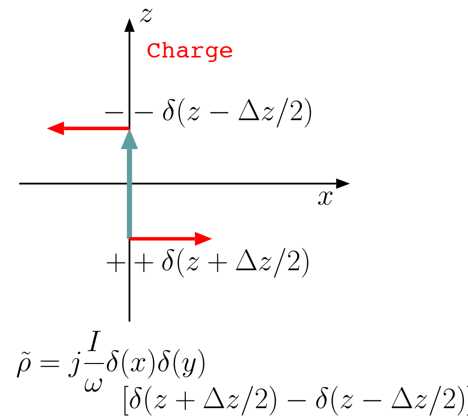
The Hertzian dipole is a *hypothetical* radiation element defined and introduced above. Its main utility is that it has the simplest radiation properties that one could imagine and use as a building block to represent more complicated (and practical rather than hypothetical) radiation elements.

- A Hertzian dipole was defined as a filament of an infinitesimal length Δz which is carrying a constant (z-independent) current at each instant of time t.

  - Since outside the filament the current vanishes, charge conservation and the continuity equation
$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$
demand that there has to be a time-varying charge accumulation at the two ends of the filament.

    Since for a $\hat{z}$ directed Hertzian dipole, $\mathbf{J} = \hat{z} J_z$, we can write the phasor domain form of the continuity equation as
$$j\omega\tilde{\rho} + \frac{\partial \tilde{J}_z}{\partial z} = 0.$$
Thus, with
$$\tilde{J}_z = I\,\delta(x)\delta(y)\,\mathrm{rect}(\frac{z}{\Delta z})$$
and
$$\frac{\partial \tilde{J}_z}{\partial z} = I\,\delta(x)\delta(y)[\delta(z + \frac{\Delta z}{2}) - \delta(z - \frac{\Delta z}{2})],$$
we get
$$\tilde{\rho} = -\frac{1}{j\omega}\frac{\partial \tilde{J}_z}{\partial z} = j\frac{I}{\omega}\delta(x)\delta(y)[\delta(z + \frac{\Delta z}{2}) - \delta(z - \frac{\Delta z}{2})].$$





Depicted charge density (red)leads the depicted current density (blue) profile by a quarter period because of j term in charge density.

Positive reservoir of charge at z<0 end of the dipole discharges into the negative reservoir at the other end causing half a cycle of z-directed current across the filament.

By the end of half-cycle the top end is positively charged and the bottom end negatively, so a new half-cycle with motions in the opposite direction starts.

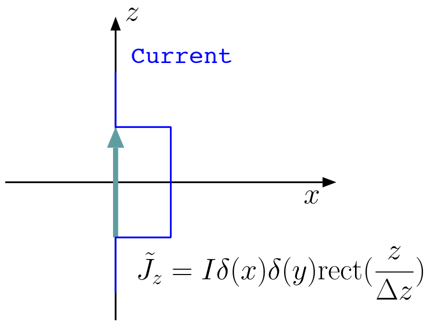
In time-domain this corresponds to

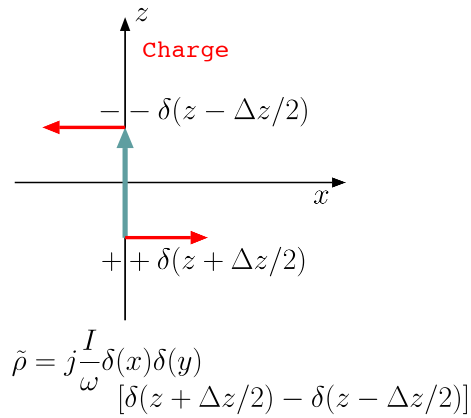
$$\rho(\mathbf{r},t) = \frac{I}{\omega}\delta(x)\delta(y)[\delta(z-\frac{\Delta z}{2}) - \delta(z+\frac{\Delta z}{2})]\sin(\omega t)\ \frac{\mathrm{C}}{\mathrm{m}^3}$$

accompanying the current density variation

$$\mathbf{J}(\mathbf{r},t) \ = \ I\delta(x)\delta(y)\mathrm{rect}(\frac{z}{\Delta z})\cos(\omega t)\hat{z}\,\frac{\mathrm{A}}{\mathrm{m}^2}.$$

- Clearly, the result above shows that the "ends" of a Hertzian dipole element located at $z = \pm\frac{\Delta z}{2}$ serve as point-charge reservoirs (of opposite polarities) sustaining the current variations of the element.
  - Radiated fields of the Hertzian dipole should be attributed to both the time-varying ρ and the time-varying $\mathbf{J}$ even though considerations of $\mathbf{J}$ will be sufficient to determine the radiated fields owing to the dependence of ρ on $\mathbf{J}$ that is built-in within Maxwell's equations.





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