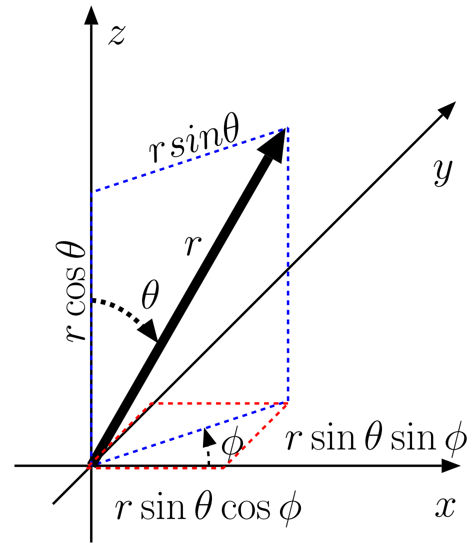
5 Vector calculus in spherical coordinates



In studies of radiation from compact antennas it is more convenient to use **spherical coordinates** instead of the Cartesian coordinates that we are familiar with. In this lecture we will learn

1. how to represent vectors and vector fields in spherical coordinates,

2. how to perform div, grad, curl, and Laplacian operations in spherical coordinates.

- A 3D position vector
$$\mathbf{r} = (x, y, z)$$
with *Cartesian coordinates* (x, y, z) is said to have *spherical coordinates* (r, θ, ϕ) where

$$\begin{aligned}
\text{length } r &\equiv |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2} \\
\text{zenith angle } \theta &= \tan^{-1}\frac{\sqrt{x^2 + y^2}}{z} \\
\text{azimuth angle } \phi &= \tan^{-1}\frac{y}{x}.
\end{aligned}$$

In terms of spherical coordinates, Cartesian coordinates can be expressed as

$$\begin{aligned}
x &= r\sin\theta\cos\phi \\
y &= r\sin\theta\sin\phi \\
z &= r\cos\theta.
\end{aligned}$$

Ratios $x/r = \sin\theta\cos\phi$, $y/r = \sin\theta\sin\phi$, and $z/r = \cos\theta$ are referred to as **direction cosines** as they represent the *cosine* of the angle between vector $\mathbf{r} = (x, y, z)$ and the x-, y-, and z-axes, respectively.

- In Cartesian coordinates we have mutually orthogonal unit vectors

$$\hat{x},\ \hat{y},\ \hat{z}$$

pointing in the direction of increasing Cartesian coordinates x, y, z, respectively.

- Likewise, in spherical coordinates we have mutually orthogonal unit vectors

$$\hat{r},\ \hat{\theta},\ \hat{\phi}$$

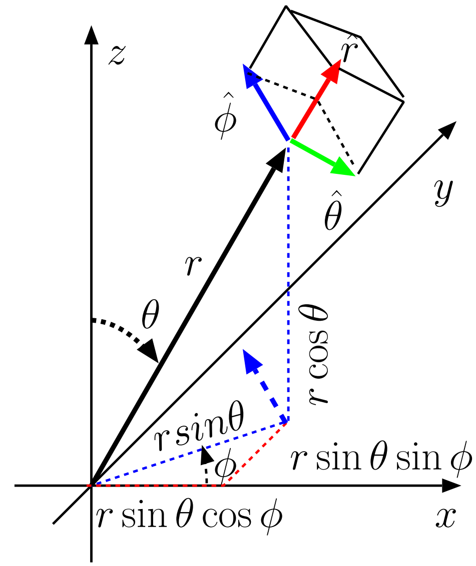
pointing in the direction of increasing coordinates r, θ, ϕ, respectively.

- However, unlike $\hat{x}$, $\hat{y}$, $\hat{z}$, the unit vectors $\hat{r}$, $\hat{\theta}$, $\hat{\phi}$ are **not global** — rather they are **local** in the sense that their directions depend on the local coordinates.

  – The local nature of $\hat{r}$, $\hat{\theta}$, $\hat{\phi}$ becomes clear when they are expressed in terms of the global unit vectors $\hat{x}$, $\hat{y}$, $\hat{z}$ as follows:

$$\begin{aligned}
\hat{r} &= \frac{\mathbf{r}}{r} = \frac{(x,y,z)}{r} = \hat{x}\sin\theta\cos\phi + \hat{y}\sin\theta\sin\phi + \hat{z}\cos\theta \\
\hat{\phi} &= \frac{(-y,x,0)}{\sqrt{x^2+y^2}} = -\hat{x}\sin\phi + \hat{y}\cos\phi \\
\hat{\theta} &= \hat{\phi}\times\hat{r} = \hat{x}\cos\theta\cos\phi + \hat{y}\cos\theta\sin\phi - \hat{z}\sin\theta
\end{aligned}$$

  Make sure you understand each of the terms above with reference to the figure shown in the margin.



Unit-vectors **$\hat{r}$, $\hat{\theta}$,and $\hat{\phi}$ shown in red, green, and blue point in mutually orthogonal directions of increasing spherical coordinates r, θ, and ϕ, respectively, such that $\hat{\theta}\times\hat{\phi}=\hat{r}$. Note that $\hat{r}$, $\hat{\theta}$,and $\hat{\phi}$ are local unit vectors (i.e., coordinate dependent) unlike the global unit vectors $\hat{x}$, $\hat{y}$, and $\hat{z}$ of the Cartesian coordinate system.**

- In Cartesian coordinates we have an infinitesimal volume element

$$dV = dxdydz$$

which is used in 3D volume integrals and often denoted as "$d^3\mathbf{r}$".

  - Note that dV is the volume of a rectangular box formed by the intersection of **constant coordinate surfaces** of two infinitesimally close points having a separation vector

$$d\mathbf{r} = \hat{x}dx + \hat{y}dy + \hat{z}dz.$$

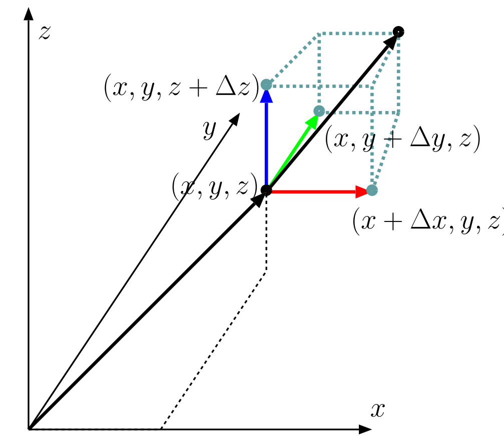
- Infinitesimal volume element $d^3\mathbf{r}$ expressed in terms of spherical coordinates and their increments is

$$dV = (dr)\,(rd\theta)\,(r\sin\theta d\phi) = r^2\sin\theta dr d\theta d\phi.$$

  - Once again dV is the volume of a rectangular box formed by the intersection of **constant coordinate surfaces** of two infinitesimally close points having a separation vector

$$d\mathbf{r} = \hat{r}dr + \hat{\theta}rd\theta + \hat{\phi}r\sin\theta d\phi.$$

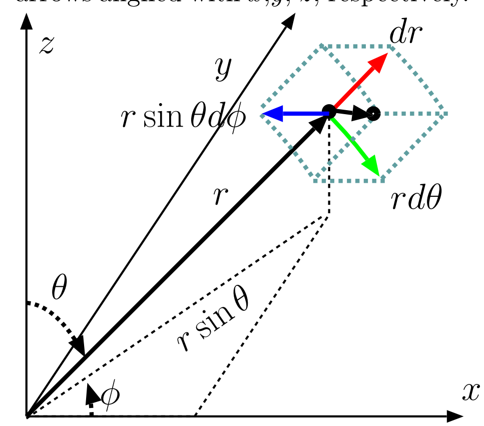
  - Note that in this case constant coordinate surfaces are no longer planar globally, but over infinitesimal dimensions of dV the surfaces will appear locally planar.



Any vector

$$\mathbf{A}(\mathbf{r}) = A_x\hat{x} + A_y\hat{y} + A_z\hat{z},$$

where A_x, A_y, and A_z are the projections of $\mathbf{A}(\mathbf{r})$ on red, green, and blue arrows aligned with $\hat{x}$,$\hat{y}$, $\hat{z}$, respectively.



Any vector

$$\mathbf{A}(\mathbf{r}) = A_r\hat{r} + A_\theta\hat{\theta} + A_\phi\hat{\phi},$$

where A_r, A_θ, and A_ϕ are the projections of $\mathbf{A}(\mathbf{r})$ on red, green, and blue arrows aligned with $\hat{r}$,$\hat{\theta}$, $\hat{\phi}$, respectively.

In Cartesian coordinates div, curl, and grad

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$\nabla V = \frac{\partial V}{\partial x}\hat{x} + \frac{\partial V}{\partial y}\hat{y} + \frac{\partial V}{\partial z}\hat{z}$$

are obtained by applying the del operator

$$\nabla \equiv (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$$

"algebraically" to vectors

$$\mathbf{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$$

and scalars

$$V(x, y, z)$$

as indicated above.

In spherical coordinates div, curl, and grad

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2}\frac{\partial(r^2 A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta A_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

$$\nabla \times \mathbf{A} = \begin{vmatrix} \frac{\hat{r}}{r^2\sin\theta} & \frac{\hat{\theta}}{r\sin\theta} & \frac{\hat{\phi}}{r} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix}$$

$$\nabla V = \frac{\partial V}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial\phi}\hat{\phi}$$

are obtained for vectors

$$\mathbf{A} = A_r\hat{r} + A_\theta\hat{\theta} + A_\phi\hat{\phi}$$

and scalars

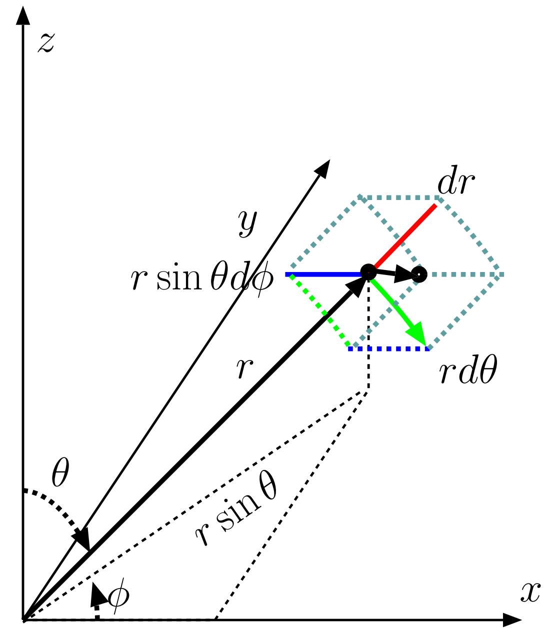
$$V(r, \theta, \phi)$$

as indicated above. **Note that there is no del operator that "works algebraically" in spherical coordinates**.

Example 1: Verify the $\hat{r}$ component of $\nabla \times \mathbf{A}$ formula in spherical coordinates by showing that it corresponds to

$$\lim_{A_C \to 0} \frac{\oint_C \mathbf{A} \cdot d\mathbf{l}}{A_C}$$

where A_C is the enclosed area of contour C orthogonal to $\hat{r}$ marked in the margin by blue and green edges.



Solution: In spherical coordinates

$$\nabla \times \mathbf{A} = \begin{vmatrix} \frac{\hat{r}}{r^2 \sin\theta} & \frac{\hat{\theta}}{r \sin\theta} & \frac{\hat{\phi}}{r} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix}$$

and, therefore, $\hat{r}$ component of $\nabla \times \mathbf{A}$ is

$$(\nabla \times \mathbf{A}) \cdot \hat{r} \;=\; \frac{1}{r^2 \sin\theta}(\frac{\partial}{\partial\theta} r\sin\theta A_\phi - \frac{\partial}{\partial\phi} rA_\theta) = \frac{1}{r\sin\theta}(\frac{\partial}{\partial\theta}\sin\theta A_\phi - \frac{\partial}{\partial\phi}A_\theta).$$

To show that this expression corresponds (as it should by definition) to

$$\lim_{A_C \to 0} \frac{\oint_C \mathbf{A} \cdot d\mathbf{l}}{A_C}$$

where circulation path C and enclosed area A_C are as described in the question statement, we first note that

$$A_C \approx (r\sin\theta d\phi)(rd\theta)$$

to second order in increments $d\theta$ and $d\phi$. Also,

$$\begin{aligned}\oint_C \mathbf{A}\cdot d\mathbf{l} &= A_\theta(r,\theta,\phi)r d\theta + A_\phi(r,\theta+d\theta,\phi) r\sin(\theta+d\theta)d\phi \\ &\quad - A_\theta(r,\theta,\phi+d\phi)r d\theta - A_\phi(r,\theta,\phi)r\sin\theta d\phi\end{aligned}$$

starting on the green edge. Thus

$$\begin{aligned}\frac{\oint_C \mathbf{A}\cdot d\mathbf{l}}{A_C} &= \frac{A_\theta(r,\theta,\phi) - A_\theta(r,\theta,\phi+d\phi)}{r\sin\theta d\phi} \\ &\quad + \frac{A_\phi(r,\theta+d\theta,\phi)\sin(\theta+d\theta) - A_\phi(r,\theta,\phi)\sin\theta}{r\sin\theta d\theta}\end{aligned}$$

which yields in the limit of vanishing $d\theta$ and $d\phi$

$$-\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}A_\theta + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\sin\theta A_\phi \equiv (\nabla\times\mathbf{A})\cdot\hat{r}$$

as requested.

See Appendix A and B in *Rao* for a complete coverage of the derivation of div, grad, curl in spherical coordinates.

Example 2: Verify the gradient procedure

$$\nabla V = \frac{\partial V}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{\phi}$$

in spherical coordinates.

Solution: Independent of the coordinate employed, the *total differential* dV and the *gradient* ∇V of a scalar field $V(\mathbf{r})$ are related by

$$dV = \nabla V \cdot d\mathbf{r}.$$

In the Cartesian coordinate system where $V = V(x, y, z)$, this relation expands as

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz = \nabla V \cdot (\hat{x}dx + \hat{y}dy + \hat{z}dz)$$

and implies

$$\nabla V = \frac{\partial V}{\partial x}\hat{x} + \frac{\partial V}{\partial y}\hat{y} + \frac{\partial V}{\partial z}\hat{z}.$$

Likewise, for spherical coordinates where $V = V(r, \theta, \phi)$, we have

$$dV = \frac{\partial V}{\partial r}dr + \frac{\partial V}{\partial \theta}d\theta + \frac{\partial V}{\partial \phi}d\phi = \nabla V \cdot (\hat{r}dr + \hat{\theta}r d\theta + \hat{\phi}r\sin\theta d\phi)$$

implying that

$$\nabla V = \frac{\partial V}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{\phi}.$$

Example 3: Show that the Laplacian of a scalar field $V(r,\theta,\phi)$ is specified as

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial V}{\partial r}) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial V}{\partial\theta}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}.$$

Solution: Since the Laplacian is the divergence of a gradient, we start by noting that

$$\nabla V = \frac{\partial V}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial\phi}\hat{\phi}.$$

Applying to this vector the divergence formula

$$\begin{aligned}
\nabla\cdot\nabla V &= \frac{1}{r^2}\frac{\partial(r^2(\nabla V)_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta(\nabla V)_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial(\nabla V)_\phi}{\partial\phi} \\
&= \frac{1}{r^2}\frac{\partial(r^2\frac{\partial V}{\partial r})}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta\frac{1}{r}\frac{\partial V}{\partial\theta})}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial(\frac{1}{r\sin\theta}\frac{\partial V}{\partial\phi})}{\partial\phi}
\end{aligned}$$

the above result for the Laplacian is readily obtained.