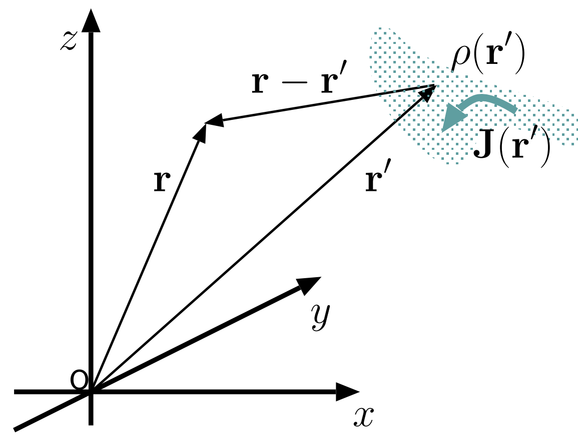
3 Lorenz gauge and inhomogeneous wave equation



Last lecture we found out that given the static sources

$$\rho = \rho(\mathbf{r}) \quad \text{and} \quad \mathbf{J} = \mathbf{J}(\mathbf{r}),$$

static fields

$$\mathbf{E} = -\nabla V \quad \text{and} \quad \mathbf{B} = \nabla \times \mathbf{A}$$

satisfying

Electrostatics: (curl-free)

$$\begin{aligned} \nabla \cdot \mathbf{D} &= \rho \\ \nabla \times \mathbf{E} &= 0 \\ \mathbf{D} &= \epsilon_o \mathbf{E} \end{aligned}$$

Magnetostatics: (divergence-free)

$$\begin{aligned} \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{H} &= \mathbf{J} \\ \mathbf{B} &= \mu_o \mathbf{H} \end{aligned}$$

can be computed using the potentials

$$V(\mathbf{r}) = \int \frac{\rho(\mathbf{r}')}{4\pi\epsilon_o|\mathbf{r}-\mathbf{r}'|} d^3\mathbf{r}',$$

the solution of

$$\nabla^2 V = -\frac{\rho}{\epsilon_o}.$$

$$\mathbf{A}(\mathbf{r}) = \int \frac{\mu_o \mathbf{J}(\mathbf{r}')}{4\pi|\mathbf{r}-\mathbf{r}'|} d^3\mathbf{r}',$$

the solution of

$$\nabla^2 \mathbf{A} = -\mu_o \mathbf{J}.$$

- Over the next two lectures we will explain why in case of time-varying sources

$$\rho = \rho(\mathbf{r}, t) \quad \text{and} \quad \mathbf{J} = \mathbf{J}(\mathbf{r}, t),$$

the full set of Maxwell's equations (see margin) can be satisfied by

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} \quad \text{and} \quad \mathbf{B} = \nabla \times \mathbf{A}$$

in terms of delayed or **retarded potentials** specified as

$$\begin{aligned} \nabla \cdot \mathbf{D} &= \rho \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \times \mathbf{H} &= \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \end{aligned}$$

$$V(\mathbf{r}, t) = \int \frac{\rho(\mathbf{r}', t - \frac{|\mathbf{r}-\mathbf{r}'|}{c})}{4\pi\epsilon_o|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}',$$

the solution of inhomogeneous wave equation

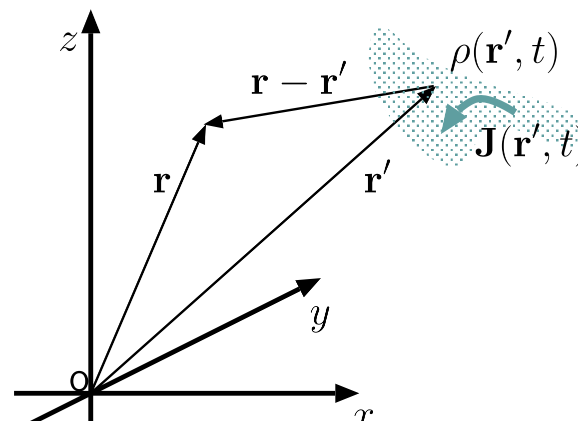
$$\nabla^2 V - \mu_o\epsilon_o \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon_o}$$

$$\mathbf{A}(\mathbf{r}, t) = \int \frac{\mu_o \mathbf{J}(\mathbf{r}', t - \frac{|\mathbf{r}-\mathbf{r}'|}{c})}{4\pi|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}',$$

the solution of inhomogeneous wave equation

$$\nabla^2 \mathbf{A} - \mu_o\epsilon_o \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_o \mathbf{J}$$

where $c \equiv \frac{1}{\sqrt{\mu_o\epsilon_o}}$ is the speed of light in free space.



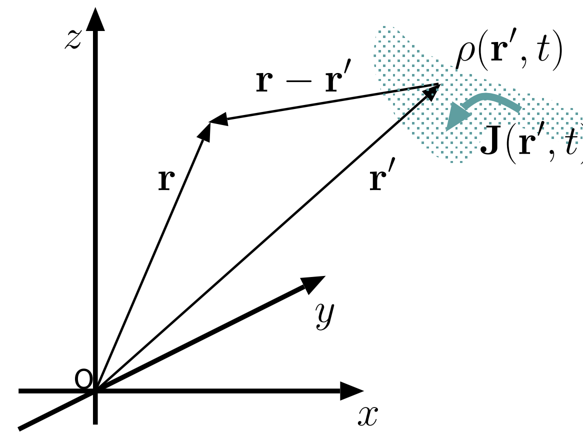
- Note that *retarded* potentials

$$V(\mathbf{r}, t) \quad \text{and} \quad \mathbf{A}(\mathbf{r}, t)$$

are essentially weighted and *delayed* sums of charge and current densities

$$\rho(\mathbf{r},t) \quad \text{and} \quad \mathbf{J}(\mathbf{r},t),$$

while the fields $\mathbf{E}$ and $\mathbf{B}$ are obtained by spatial and temporal derivatives of the potentials.



- Alternatively, we can first use

$$\mathbf{A}(\mathbf{r},t) = \int \frac{\mu_o \mathbf{J}(\mathbf{r}', t - \frac{|\mathbf{r}-\mathbf{r}'|}{c})}{4\pi|\mathbf{r}-\mathbf{r}'|} d^3\mathbf{r}' \;\; \Rightarrow \;\; \mathbf{B} = \nabla \times \mathbf{A}$$

and then find the anti-derivative of Ampere's law

$$\nabla \times \frac{\mathbf{B}}{\mu_o} = \epsilon_o \frac{\partial \mathbf{E}}{\partial t}$$

to determine $\mathbf{E}$ outside the region where $\mathbf{J}$ is non-zero, bypassing the use of scalar retarded potential $V(\mathbf{r},t)$ — that is the most common approach used in **radiation studies**.

We will next verify the procedure outlined above and the start discussing its applications in radiation studies.

- The full set of Maxwell's equations is repeated in the margin for convenience. Divergence-free nature of $\mathbf{B}$ compels us to define a vector potential $\mathbf{A}$ via

$$\mathbf{B} = \nabla \times \mathbf{A}$$

just as before. Inserting this in Faraday's law we get

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\nabla \times \mathbf{A} \quad \Rightarrow \quad \nabla \times (\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}) = 0.$$

Evidently

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \text{ is curl free, so it must be true that } \mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla V,$$

or

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}$$

in terms of some scalar potential V.

$$\begin{aligned} \nabla \cdot \mathbf{D} &= \rho \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \times \mathbf{H} &= \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \end{aligned}$$

Main difference from statics appears to be the need for *two* potentials, instead of one, to represent the electric field E under time-varying conditions. We continue ....

- Now substitute

$$\mathbf{B} = \nabla \times \mathbf{A} \text{ and } \mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}$$

in the remaining two Maxwell's equations — Gauss's and Ampere's laws

$$\nabla \cdot (\epsilon_o \mathbf{E}) = \rho \quad \text{and} \quad \nabla \times (\mu_o^{-1} \mathbf{B}) = \mathbf{J} + \frac{\partial}{\partial t}(\epsilon_o \mathbf{E}),$$

that we have not touched yet. Upon substitutions we get

$$\epsilon_o \nabla \cdot (-\nabla V - \frac{\partial \mathbf{A}}{\partial t}) \;=\; \rho \quad \text{and} \quad \underbrace{\nabla \times \nabla \times \mathbf{A}}_{\nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}} = \mu_o \mathbf{J} + \mu_o \epsilon_o \frac{\partial}{\partial t}(-\nabla V - \frac{\partial \mathbf{A}}{\partial t}),$$

which looks like a big mess.

- But if we specify

$$\nabla \cdot \mathbf{A} = -\mu_o \epsilon_o \frac{\partial V}{\partial t} \qquad (\textbf{Lorenz gauge})$$

these messy equations simplify as

$$\nabla^2 V - \mu_o \epsilon_o \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon_o} \quad \text{and} \quad \nabla^2 \mathbf{A} - \mu_o \epsilon_o \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_o \mathbf{J}$$

which we recognize as the inhomogeneous or "forced" wave equations for V and $\mathbf{A}$ stated earlier on.

  - The derivation of the decoupled wave equations above hinged upon our use of **Lorenz gauge** which reduces to the **Coulomb gauge**, $\nabla \cdot \mathbf{A} = 0$, in static situations.
  - Note also that the forced wave equations reduce to Poisson's equations under time-static conditions.

  - Since we know how to solve the unforced wave equation from ECE 329, and since we know how to solve the Poisson's equation, it is now a matter of combining those methods to solve the forced wave equations obtained above.

Just a few additional comments on ***gauge selection*** before we go on (next lecture):

- Gauge selection amounts to deciding what to assign to $\nabla \cdot \mathbf{A}$.
- We can make any assignment that pleases us. This is like choosing the ground node in a circuit problem. Whatever simplifies the problem the most is the best gauge to use.
  - Lorenz gauge is clearly a good one since it led to decoupled wave equations which are very convenient to work with.

  We can attack the decoupled equations for V and $\mathbf{A}$ *one at a time*.