2 Static fields and potentials

Static fields

$$\mathbf{E} = \mathbf{E}(\mathbf{r}), \;\; \mathbf{D} = \mathbf{D}(\mathbf{r}), \;\; \mathbf{B} = \mathbf{B}(\mathbf{r}), \;\; \mathbf{H} = \mathbf{H}(\mathbf{r})$$

independent of the time variable t are produced by static source distributions

$$\rho = \rho(\mathbf{r}) \;\; \text{and} \;\; \mathbf{J} = \mathbf{J}(\mathbf{r})$$

which only depend on position vector $\mathbf{r} = (x, y, z)$. In case of static fields Maxwell's equations simplify and decouple as

Time-dependent:

$$\begin{aligned} \nabla \cdot \mathbf{D} &= \rho \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \times \mathbf{H} &= \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \end{aligned}$$

$$\overset{\Rightarrow}{\frac{\partial}{\partial t} = 0}$$

Electrostatics: (curl-free)

$$\begin{aligned} \nabla \cdot \mathbf{D} &= \rho \\ \nabla \times \mathbf{E} &= 0 \\ \mathbf{D} &= \epsilon_o \mathbf{E} \end{aligned}$$

Magnetostatics: (divergence-free, solenoidal)

$$\begin{aligned} \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{H} &= \mathbf{J} \\ \mathbf{B} &= \mu_o \mathbf{H} \end{aligned}$$

Important vector identities:

- $\nabla \times (\nabla V) = 0$
- $\nabla \cdot (\nabla \times \mathbf{A}) = 0$
- $\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$.

Electrostatics: (curl-free)

$$\begin{aligned} \nabla \cdot \mathbf{D} &= \rho \\ \nabla \times \mathbf{E} &= 0 \\ \mathbf{D} &= \epsilon_o \mathbf{E} \end{aligned}$$

Since all curl-free fields can be expressed in terms of a scalar gradient, we choose

$$\mathbf{E} = -\nabla V,$$

where

$$V = V(x, y, z)$$

is called **electrostatic potential**.

Magnetostatics: (divergence-free)

$$\begin{aligned} \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{H} &= \mathbf{J} \\ \mathbf{B} &= \mu_o \mathbf{H} \end{aligned}$$

Since all divergence-free fields can be expressed in terms of a curl, we choose

$$\mathbf{B} = \nabla \times \mathbf{A}$$

where

$$\mathbf{A} = \mathbf{A}(x, y, z)$$

is called **vector potential**.

Electrostatics: (curl-free)

$$\nabla \cdot \mathbf{D} = \rho$$
$$\nabla \times \mathbf{E} = 0$$
$$\mathbf{D} = \epsilon_o \mathbf{E}$$

such that

$$\mathbf{E} = -\nabla V.$$

Electrostatic potential

$$V = V(x, y, z)$$

signifies the kinetic energy available (i.e., stored potential energy) — total energy being $\frac{1}{2} m\mathbf{v} \cdot \mathbf{v} + qV$ — per unit charge in a static field measured from a convenient reference point (ground).

Magnetostatics: (divergence-free)

$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{H} = \mathbf{J}$$
$$\mathbf{B} = \mu_o \mathbf{H}$$

such that

$$\mathbf{B} = \nabla \times \mathbf{A}.$$

If we apply the constraint $\nabla \cdot \mathbf{A} = 0$ — known as **Coulomb gauge** and discussed in more detail next lecture — then the **vector potential**

$$\mathbf{A} = \mathbf{A}(x, y, z)$$

can be interpreted as kinetic momentum $m\mathbf{v}$ available — total (canonical) momentum being $m\mathbf{v} + q\mathbf{A}$ — per unit charge in a static field.

- In general, given V and $\mathbf{A}$, it is easy to compute $\mathbf{E}$ and $\mathbf{B}$.
- **How do we get V and $\mathbf{A}$ (and thus $\mathbf{E}$ and $\mathbf{B}$) from ρ and $\mathbf{J}$?** Before addressing this question in full generality let's review the electric field $\mathbf{E}$ and the electrostatic potential V of a stationary point charge.

Coulomb's law specifies the electric field of a stationary charge Q at the origin as

$$\mathbf{E}(\mathbf{r}) = \frac{Q}{4\pi\epsilon_o|\mathbf{r}|^2}\hat{r}$$

as a function of position vector $\mathbf{r} = (x, y, z)$ with a magnitude

$$|\mathbf{r}| \equiv r = \sqrt{x^2 + y^2 + z^2} \quad \text{and direction unit vector} \quad \hat{r} = \frac{\mathbf{r}}{r}.$$

- This Coulomb field $\mathbf{E}(\mathbf{r})$ will exert a force $\mathbf{F} = q\mathbf{E}(\mathbf{r})$ on any stationary "test charge" q brought within distance r of Q (see margin).
- The associated electrostatic potential is

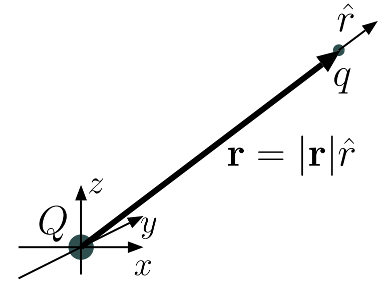
$$V(\mathbf{r}) = \frac{Q}{4\pi\epsilon_o|\mathbf{r}|}$$

with an implied ground for $|\mathbf{r}| \to \infty$.

Verification: this can be done in two ways,

1. by computing $-\nabla V \equiv \mathbf{E}(\mathbf{r})$, or
2. by computing the line integral $\int_{\mathbf{r}}^{\infty} \mathbf{E} \cdot d\mathbf{l} \equiv V(\mathbf{r})$ along any path.

In HW 1 we will ask you to verify the potential of the point charge using both methods.



Force exerted by Q on q:

$$\mathbf{F} = q\mathbf{E}$$

with electric field

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_o|\mathbf{r}|^2}\hat{r}$$

With multiple Q's superpose multiple E's

Poisson's equations:

Electrostatics: Since

$$\nabla \times \mathbf{E} = 0 \quad \Rightarrow \quad \mathbf{E} = -\nabla V,$$

we have

$$\mathbf{D} = \epsilon_o \mathbf{E} \;\; \text{and} \;\; \nabla \cdot \mathbf{D} = \rho$$

implying

$$\nabla \cdot (-\epsilon_o \nabla V) = \rho \quad \Rightarrow \nabla^2 V = -\frac{\rho}{\epsilon_o}.$$

Magnetostatics: Since

$$\nabla \cdot \mathbf{B} = 0 \quad \Rightarrow \quad \mathbf{B} = \nabla \times \mathbf{A},$$

we have

$$\mathbf{B} = \mu_o \mathbf{H} \;\; \text{and} \;\; \nabla \times \mathbf{H} = \mathbf{J}$$

implying

$$\nabla \times (\mu_o^{-1} \nabla \times \mathbf{A}) = \mathbf{J} \quad \Rightarrow \nabla^2 \mathbf{A} = -\mu_o \mathbf{J}$$

after using

$$\nabla \cdot \mathbf{A} = 0 \quad (\textbf{Coulomb gauge})$$

in the expansion of

$$\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}.$$

- We can get V and $\mathbf{A}$ from ρ and $\mathbf{J}$ by solving the **Poisson's equations**

$$\nabla^2 V = -\frac{\rho}{\epsilon_o} \quad \text{and} \quad \nabla^2 \mathbf{A} = -\mu_o \mathbf{J}$$

where

$$\nabla^2 \equiv \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \;\; \text{is } \textbf{Laplacian} \text{ operator.}$$

The solution of **electrostatic Poisson's equation**

$$\nabla^2 V = -\frac{\rho}{\epsilon_o}$$

with an arbitrary $\rho(\mathbf{r})$ existing over any finite region in space can be obtained as

$$V(\mathbf{r}) = \int \frac{\rho(\mathbf{r}')}{4\pi\epsilon_o|\mathbf{r}-\mathbf{r}'|} d^3\mathbf{r}'$$

where $d^3\mathbf{r}' \equiv dx'dy'dz'$ and the 3D volume integral on the right over the primed coordinates is performed over the entire region where the charge density is non-zero (see margin).

- **Verification:** The solution above can be verified by combining a number of results we have seen earlier on:

  1. Electric potential $V(\mathbf{r})$ of a point charge Q at the origin is

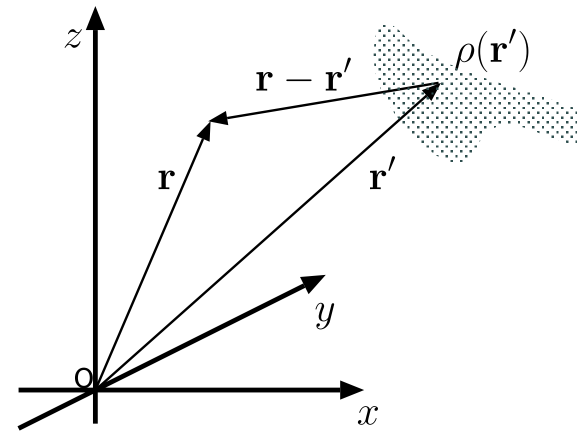
     $$V(\mathbf{r}) = \frac{Q}{4\pi\epsilon_o|\mathbf{r}|}.$$

     Clearly, this *singular* result is a solution of Poisson's equation above for a charge density input of

     $$\rho(\mathbf{r}) = Q\delta(\mathbf{r}).$$

     (a) Using ECE 210-like terminology and notation, the above result can be represented as

     $$\delta(\mathbf{r}) \rightarrow \boxed{\text{Poisson's Eqn}} \rightarrow \frac{1}{4\pi\epsilon_o|\mathbf{r}|}$$



The general solution

$$V(x,y,z)$$

is obtained by performing a 3D volume integral of

$$\frac{\rho(x',y',z')}{4\pi\epsilon_o|(x,y,z)-(x',y',z')|}$$

over the primed coordinates. In abbreviated notation

$$d^3\mathbf{r}' \equiv dx'dy'dz'$$

denotes an infinitesimal volume of the primed coordinate system.

identifying the output on the right as a 3D "impulse response" of the **linear** and **shift-invariant** (LSI) system represented by the Poisson's equation.

(b) Because of shift-invariance, we have

$$\delta(\mathbf{r}-\mathbf{r}') \rightarrow \boxed{\text{Poisson's Eqn}} \rightarrow \frac{1}{4\pi\epsilon_o|\mathbf{r}-\mathbf{r}'|},$$

meaning that a shifted impulse causes a shifted impulse response.

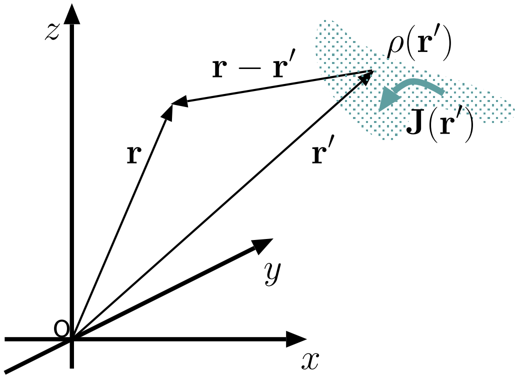
The shifted impulse response is usually called "Green's function" $G(\mathbf{r},\mathbf{r}')$ in EM theory.

(c) Because of linearity, we are allowed to use superpositioning arguments like

$$\int \rho(\mathbf{r}')\delta(\mathbf{r}-\mathbf{r}')d^3\mathbf{r}' = \rho(\mathbf{r}) \rightarrow \boxed{\text{Poisson's Eqn}} \rightarrow \int \rho(\mathbf{r}')\frac{1}{4\pi\epsilon_o|\mathbf{r}-\mathbf{r}'|}d^3\mathbf{r}' = V(\mathbf{r}),$$

which concludes our verification. Note how we made use of the *sifting property* of the impulse (from ECE 210) in above calculation.

Solutions of Poisson's equations:



Electrostatics:

$$\nabla^2 V = -\frac{\rho}{\epsilon_o}$$

implies a general solution

$$V(\mathbf{r}) = \int \frac{\rho(\mathbf{r}')}{4\pi\epsilon_o|\mathbf{r}-\mathbf{r}'|} d^3\mathbf{r}'.$$

Magnetostatics:

$$\nabla^2 \mathbf{A} = -\mu_o \mathbf{J}$$

implies a general solution

$$\mathbf{A}(\mathbf{r}) = \int \frac{\mu_o \mathbf{J}(\mathbf{r}')}{4\pi|\mathbf{r}-\mathbf{r}'|} d^3\mathbf{r}'.$$

These results indicate that potentials

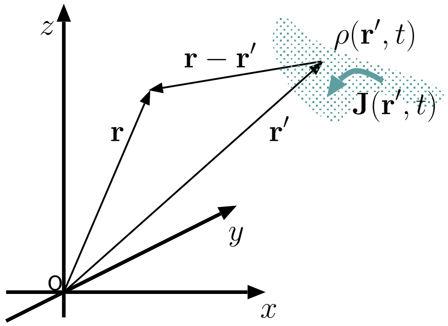
$$V(\mathbf{r}) \quad \text{and} \quad \mathbf{A}(\mathbf{r})$$

are appropriately weighted sums of

$$\rho(\mathbf{r}) \quad \text{and} \quad \mathbf{J}(\mathbf{r})$$

in convolution-like 3D space integrals.

Quasi-static approximation:



Electro-quasi-statics:

$$\mathbf{E} \approx -\nabla V$$

with

$$V(\mathbf{r}, t) \approx \int \frac{\rho(\mathbf{r}', t)}{4\pi\epsilon_o |\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}'$$

for slowly varying $\rho(\mathbf{r}, t)$.

Magneto-quasi-statics:

$$\mathbf{B} \approx \nabla \times \mathbf{A}$$

with

$$\mathbf{A}(\mathbf{r}, t) \approx \int \frac{\mu_o \mathbf{J}(\mathbf{r}', t)}{4\pi |\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}'$$

for slowly varying $\mathbf{J}(\mathbf{r}, t)$.

Validity of these quasi-static results requires that

$$T \gg \frac{L}{c}$$

where T is the period of the highest frequency in source functions $\rho(\mathbf{r}, t)$ and $\mathbf{J}(\mathbf{r}, t)$, while L is the size of the region around the source region where quasi-static approximation is acceptable. This condition cannot be satisfied as $L \to \infty$, in which case the required "fix" is to replace the potential functions above by their "retarded potential" counterparts — see next lecture.