1 Overview, Maxwell's equations

- **ECE 329** introduced the **Maxwell's equations** and examined their circuit implications (inductance, capacitance) and TEM plane-wave solutions in homogeneous media and on "two-wire" transmission lines.

- In **ECE 350** we continue our study of the solutions and applications of Maxwell's equations with a focus on:

  1. **Radiation** of *spherical* TEM waves from practical compact **antennas** (e.g., used in cell phones and wireless links).
  2. **Propagation**, **reflection**, and **interference** of TEM waves in 3D geometries.
  3. **Dispersion** effects in frequency dependent propagation media.
  4. **Guided waves** in TEM, TE, and TM modes.
  5. Fields and fluctuations in enclosed **cavities**.
  6. **Antenna reception** and link budgets in communication applications.

ECE 350 completes the introductory description of electromagnetic (EM) effects in our curriculum and prepares the student for specialization courses in EM (ECE 447, 452, 453, 454, 455, 457, 458, etc.) and applications.

Maxwell's equations:

$$\begin{aligned}\nabla\cdot\mathbf{E} &= \frac{\rho}{\epsilon_o}\\ \nabla\cdot\mathbf{B} &= 0\\ \nabla\times\mathbf{E} &= -\frac{\partial\mathbf{B}}{\partial t}\\ \nabla\times\mathbf{B} &= \mu_o\mathbf{J}+\mu_o\epsilon_o\frac{\partial\mathbf{E}}{\partial t}.\end{aligned}$$

such that

$$\mathbf{F}=q(\mathbf{E}+\mathbf{v}\times\mathbf{B}),$$

with

$$\mu_o\equiv 4\pi\times10^{-7}\,\frac{\mathbf{H}}{\mathbf{m}},$$

and

$$\epsilon_o=\frac{1}{\mu_o c^2}\approx\frac{1}{36\pi\times10^9}\,\frac{\mathbf{F}}{\mathbf{m}},$$

in mksA units, where

$$c=\frac{1}{\sqrt{\mu_o\epsilon_o}}\approx 3\times10^8\,\frac{\mathbf{m}}{\mathbf{s}}$$

is the speed of light in free space.

(In Gaussian-cgs units $\frac{\mathbf{B}}{c}$ is used in place of B above, while $\epsilon_o=\frac{1}{4\pi}$ and $\mu_o=\frac{1}{\epsilon_o c^2}=\frac{4\pi}{c^2}$.)

Review:

Maxwell's Equations:

$$\begin{aligned} \nabla \cdot \mathbf{D} &= \rho && \text{Gauss's law} \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} && \text{Faraday's law} \\ \nabla \times \mathbf{H} &= \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} && \text{Ampere's law} \end{aligned}$$

where $\Rightarrow$

Microscopic applications:

- ρ and $\mathbf{J}$ describe compact (pointlike) sources,
- $\mathbf{D} = \epsilon_o \mathbf{E}$ and $\mathbf{B} = \mu_o \mathbf{H}$

Macroscopic applications:

- ρ and $\mathbf{J}$ describe smooth sources composed of free charge carriers,
- $\mathbf{D} = \epsilon \mathbf{E}$ and $\mathbf{B} = \mu \mathbf{H}$

  specified in the in frequency domain with ω dependent

  - ***permittivity*** ϵ and
  - ***permeability*** μ.

- Fields $\mathbf{E}$ and $\mathbf{B}$ determine how a "test charge" q with mass m, position $\mathbf{r}$, and velocity $\mathbf{v} \equiv \dot{\mathbf{r}} = \frac{d\mathbf{r}}{dt}$ *accelerates* in accordance with

  Lorentz force

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

and Newton's 2nd law

$$\mathbf{F} = \frac{d}{dt} m\mathbf{v}.$$

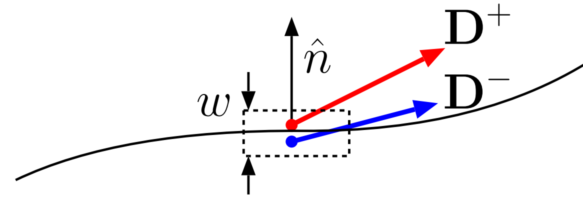
Maxwell's Equations:

$$\nabla \cdot \mathbf{D} = \rho$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

Boundary Conditions:

$$\hat{n} \cdot [\mathbf{D}^+ - \mathbf{D}^-] = \rho_s$$
$$\hat{n} \cdot [\mathbf{B}^+ - \mathbf{B}^-] = 0$$
$$\hat{n} \times [\mathbf{E}^+ - \mathbf{E}^-] = 0$$
$$\hat{n} \times [\mathbf{H}^+ - \mathbf{H}^-] = \mathbf{J}_s$$

where $\hat{n}$ is a unit normal to the boundary surface pointing from $-$ to $+$ side.



Units in mksA system:

- q[=]C=sA,
- ρ[=]C/m^3,
- $\mathbf{J}$[=]A/m^2,
- $\mathbf{E}$[=]N/C=V/m,
- $\mathbf{D}$[=]C/m^2[=]ρ_s,
- $\mathbf{B}$[=]V.s/m^2 =Wb/m^2=T,
- $\mathbf{H}$[=]A/m[=]$\mathbf{J}_s$

where C, N, V, Wb, and T are abbreviations for *Coulombs*, *Newtons*, *Volts*, *Webers*, and *Teslas*, respectively.

Charge q is quantized in units of $e = 1.602 \times 10^{-19}$ C,

a relativistic invariant.

- **Note:** the same units for
  - Displacement $\mathbf{D}$ and surface charge density ρ_s,
  - Magnetic field intensity $\mathbf{H}$ and surface current density $\mathbf{J}_s$.

- In right-handed Cartesian coordinates **div, grad**, and **curl** are produced by applying the del operator

$$\nabla \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

on vector or scalar fields as appropriate.

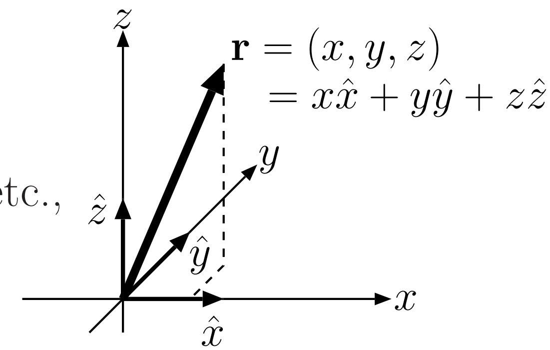
- Vectors and vector functions can be expressed in terms of mutually ***orthogonal* unit vectors** $\hat{x}$, $\hat{y}$, and $\hat{z}$ as in

  $$\mathbf{r} = (x, y, z) = x\hat{x}+y\hat{y}+z\hat{z} \quad \text{and} \quad \mathbf{E} = (E_x, E_y, E_z) = E_x\hat{x}+E_y\hat{y}+E_z\hat{z} \quad \text{etc.},$$

  where

  – $|\mathbf{r}| \equiv \sqrt{x^2+y^2+z^2}$ and $|\mathbf{E}| \equiv \sqrt{E_x^2+E_y^2+E_z^2}$ etc., are **vector magnitudes**,

  – $\hat{r} \equiv \frac{\mathbf{r}}{|\mathbf{r}|}$ and $\hat{E} \equiv \frac{\mathbf{E}}{|\mathbf{E}|}$ etc., are associated **unit vectors**, with



UNIT VECTORS AND A POSITION VECTOR IN RIGHT-HANDED CARETESIAN COORDINATES

Right handed convention: cross product vector points in the direction indicated by the thumb of your ***right hand*** when you rotate your fingers from vector **A** toward vector **B** through angle θ you decide to use.

***Dot products*:**

- $\hat{r}\cdot\hat{r} = 1$, $\hat{E}\cdot\hat{E} = 1$, $\hat{x}\cdot\hat{x} = 1$, etc.,

  but

- $\hat{x}\cdot\hat{y} = \hat{x}\cdot\hat{z} = \hat{y}\cdot\hat{z} = 0$.

Dot product $\mathbf{A}\cdot\mathbf{B}$ is a ***scalar*** which is the product of $|\mathbf{A}|$ and $|\mathbf{B}|$ and the cosine of angle θ between **A** and **B**.

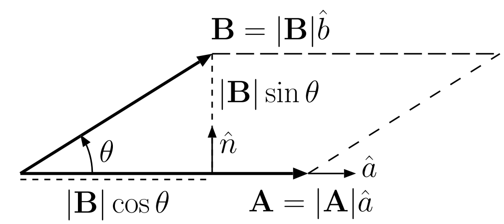
Dot product is zero when angle θ is $90°$, as in the case of $\hat{x}$ and $\hat{y}$, etc.

Cross products:

- $\hat{x}\times\hat{y} = \hat{z}$,
  $\hat{y}\times\hat{z} = \hat{x}$,
  $\hat{z}\times\hat{x} = \hat{y}$ in a **right-handed** system.

Cross product $\mathbf{A}\times\mathbf{B}$ is a ***vector*** with a magnitude the product of $|\mathbf{A}|$ and $|\mathbf{B}|$ and the sine of angle θ between **A** and **B** and a direction orthogonal to **A** and **B** in a **right-handed** sense.

Cross product is zero when the vectors cross multiplied are collinear ($\theta = 0°$) or anti-linear ($\theta = 180°$).



$\mathbf{A}\cdot\mathbf{B} = |\mathbf{A}||\mathbf{B}|\cos\theta$
DOT PRODUCT:product of projected vector lengths

$\mathbf{A}\times\mathbf{B} = |\mathbf{A}||\mathbf{B}|\sin\theta\,\hat{a}\times\hat{n}$
CROSS PRODUCT: right-handed perpendicular area vector of the parallelogram formed by co-planar vectors

Having three non-collinear force measurements $\mathbf{F}_i$ corresponding to three distinct test particle velocities $\mathbf{v}_i$ is sufficient to determine the fields E and B at any location in space produced by distant sources as illustrated by this example.

Example 1: A particle with charge $q = 1$ C passing through the origin $\mathbf{r} = (x, y, z) = 0$ of the lab frame is observed to accelerate with forces

$$\mathbf{F}_1 = 2\hat{x}, \quad \mathbf{F}_2 = 2\hat{x} - 6\hat{z}, \quad \mathbf{F}_3 = 2\hat{x} + 9\hat{y}\,\text{N}$$

when the velocity of the particle is

$$\mathbf{v}_1 = 0, \quad \mathbf{v}_2 = 2\hat{y}, \quad \mathbf{v}_3 = 3\hat{z}\,\frac{\text{m}}{\text{s}},$$

in turns. Use the Lorentz force equation

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

to determine the fields $\mathbf{E}$ and $\mathbf{B}$ at the origin.

Solution: Using the Lorentz force formula first with $\mathbf{F} = \mathbf{F}_1$ and $\mathbf{v} = \mathbf{v}_1$, we note that

$$2\hat{x} = (1)(\mathbf{E} + 0 \times \mathbf{B}),$$

which implies that

$$\mathbf{E} = 2\hat{x}\,\frac{\text{N}}{\text{C}} = 2\hat{x}\,\frac{\text{V}}{\text{m}}.$$

Next, we use

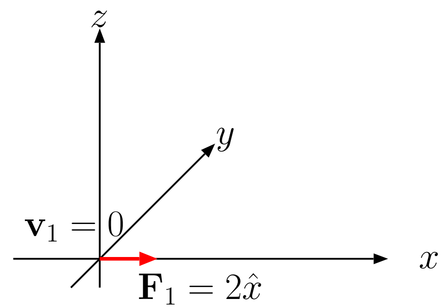
$$\mathbf{v} \times \mathbf{B} = \frac{\mathbf{F}}{q} - \mathbf{E} = \frac{\mathbf{F}}{q} - 2\hat{x}$$

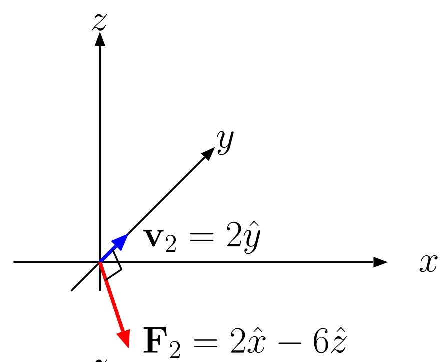
with $\mathbf{F}_2 = 2\hat{x} - 6\hat{z}$ and $\mathbf{v}_2 = 2\hat{y}$, as well as $\mathbf{E} = 2\hat{x}$ V/m, to obtain

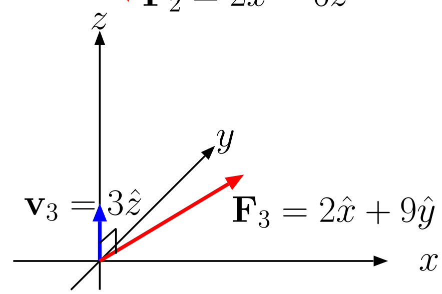
$$2\hat{y} \times \mathbf{B} = -6\hat{z} \quad \Rightarrow \quad \hat{y} \times \mathbf{B} = -3\hat{z};$$

likewise, with $\mathbf{F}_3 = 2\hat{x} + 9\hat{y}$ and $\mathbf{v}_3 = 3\hat{z}$,

$$3\hat{z} \times \mathbf{B} = 9\hat{y} \quad \Rightarrow \quad \hat{z} \times \mathbf{B} = 3\hat{y}.$$







Substitute $\mathbf{B} = B_x\hat{x} + B_y\hat{y} + B_z\hat{z}$ in above relations to obtain

$$\hat{y} \times (B_x\hat{x} + B_y\hat{y} + B_z\hat{z}) = -B_x\hat{z} + B_z\hat{x} = -3\hat{z}$$

and

$$\hat{z} \times (B_x\hat{x} + B_y\hat{y} + B_z\hat{z}) = B_x\hat{y} - B_y\hat{x} = 3\hat{y}.$$

Matching the coefficients of $\hat{x}$, $\hat{y}$, and $\hat{z}$ in each of these relations we find that

$$B_x = 3\,\frac{\text{Wb}}{\text{m}^2}, \text{ and } B_y = B_z = 0.$$

Hence, vector

$$\mathbf{B} = 3\hat{x}\,\frac{\text{Wb}}{\text{m}^2}.$$

Conservation laws

- In HW1 you are asked to derive the **continuity equation**

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$

  by taking the divergence of Ampere's Law and combining it with Gauss' Law.

  – This equation expresses the *conservation of electrical charge* by putting a constraint on charge density ρ and current density $\mathbf{J}$ as it was first explained in ECE 329 (this is just a review, recall).

- Another conservation law derived in ECE 329 from Maxwell's equations was **Poynting Theorem**, namely

$$\frac{\partial w}{\partial t} + \nabla \cdot \mathbf{S} = -\mathbf{J} \cdot \mathbf{E},$$

  where

$$w = \frac{1}{2}\epsilon_o \mathbf{E} \cdot \mathbf{E} + \frac{1}{2}\mu_o \mathbf{H} \cdot \mathbf{H} \quad \text{EM energy density,}$$

$$\mathbf{S} \equiv \mathbf{E} \times \mathbf{H} \quad \text{Poynting vector,}$$

$$-\mathbf{J} \cdot \mathbf{E} \quad \text{power produced per unit volume,}$$

  – expressing the *conservation of electromagnetic energy.*

- All conservation laws found in nature can be expressed mathematically in the forms given above in terms of a time-derivative of the volumetric density of the conserved quantity, the divergence of the flux of the conserved quantity (the so-called transport term), and a production term on the right (zero in case of charge conservation).

- The above conservation laws account for the increase/decrease of the conserved quantity density in terms of *local* transport and production effects. Hence charge conservation, for instance, is a *local* conservation principle.

  – If charge density decreases at a location, it will increase at a neighboring location because of local transport between the locations — charge cannot disappear in one volume and appear simultaneously in another volume (satisfying a so-called *global* conservation principle) without having traveled between the volumes.

  – *All conservation laws* observed in nature are *local* (as opposed to *global*) in the sense just described — the proof for this very broad statement can be based on the principle of relativity[1].

[1]Note that if charge could travel between the volumes with an infinite speed, then "global conservation" as opposed to "local conservation" could have been a viable idea — however no object can travel faster than light according to the principle of relativity and thus conservation laws have to be necessarily local and have mathematical expressions similar to those given in the *continuity equation*. A more general (but simple) proof of the local nature of *all* conservation laws (based on special relativity) is given by *Feynman* (see "The character of physical law", 1965, MIT Press).