APPENDIX A

Some Useful Mathematical Formulas

A.1 Useful Vector Identities

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}), \tag{A.1}$$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}), \tag{A.2}$$

$$\nabla \times \nabla \psi = 0, \tag{A.3}$$

$$\nabla \cdot \nabla \times \mathbf{A} = 0, \tag{a.4}$$

$$\nabla \cdot (\psi \mathbf{A}) = \mathbf{A} \cdot \nabla \psi + \psi \nabla \cdot \mathbf{A}, \tag{A.5}$$

$$\nabla \times (\psi \mathbf{A}) = \nabla \psi \times \mathbf{A} + \psi \nabla \times \mathbf{A}, \tag{A.6}$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \mathbf{B}, \tag{A.7}$$

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} + \mathbf{A} \times \nabla \times \mathbf{B} + \mathbf{B} \times \nabla \times \mathbf{A}, \tag{A.8}$$

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}\nabla \cdot \mathbf{B} - \mathbf{B}\nabla \cdot \mathbf{A}, \tag{A.9}$$

$$\nabla \times \nabla \times \mathbf{A} = \nabla\nabla \cdot \mathbf{A} - \nabla^2 \mathbf{A}. \tag{A.10}$$

In Cartesian coordinates, $\nabla^2 \mathbf{A}$ can be decomposed as

$$\nabla^2 \mathbf{A} = \hat{x}\nabla^2 A_x + \hat{y}\nabla^2 A_y + \hat{z}\nabla^2 A_z, \tag{A.11}$$

because ∇^2 commutes with $\hat{x}$, $\hat{y}$, and $\hat{z}$, i.e., $\nabla^2 \hat{x} = \hat{x}\nabla^2$ and so on. This is not true in other curvilinear coordinates; hence, this decomposition is not allowed.

A.2 Gradient, Divergence, Curl, and Laplacian in Rectangular, Cylindrical, Spherical, and General Orthogonal Curvilinear Coordinate Systems

(a) Rectangular System; x, y, z:

$$\nabla \psi = \frac{\partial \psi}{\partial x}\hat{x} + \frac{\partial \psi}{\partial y}\hat{y} + \frac{\partial \psi}{\partial z}\hat{z}, \tag{A.12}$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}, \tag{A.13}$$

$$\nabla \times \mathbf{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{x} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{y} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{z}, \tag{A.14}$$

$$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2}. \tag{A.15}$$

(b) Cylindrical System; ρ, ϕ, z:

$$\nabla\psi = \frac{\partial\psi}{\partial\rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial\psi}{\partial\phi}\hat{\phi} + \frac{\partial\psi}{\partial z}\hat{z}, \tag{A.16}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}, \tag{A.17}$$

$$\nabla\times\mathbf{A} = \left(\frac{1}{\rho}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right)\hat{\rho} + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho}\right)\hat{\phi} + \frac{1}{\rho}\left(\frac{\partial}{\partial\rho}(\rho A_\phi) - \frac{\partial A_\rho}{\partial\phi}\right)\hat{z}, \tag{A.18}$$

$$\nabla^2\psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}. \tag{A.19}$$

(c) Spherical System; r, θ, ϕ:

$$\nabla\psi = \frac{\partial\psi}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}\hat{\phi}, \tag{A.20}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}, \tag{A.21}$$

$$\nabla\times\mathbf{A} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \frac{\partial A_\theta}{\partial\phi}\right]\hat{r} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}(rA_\phi)\right]\hat{\theta} + \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right]\hat{\phi}, \tag{A.22}$$

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}. \tag{A.23}$$

(d) General Orthogonal Curvilinear Coordinate System; x_1, x_2, x_3:

The metric coefficients (h_1, h_2, h_3) in a general orthogonal curvilinear coordinate system are defined by

$$ds_i = h_i\,dx_i; \quad i = 1 \text{ or } 2, \text{ or } 3, \tag{A.24}$$

where ds_i denotes a differential length in the direction of dx_i. Moreover, the variable, x_i may not have the dimension of length. One way of finding the metric coefficients is to express the rectangular variables in terms of the variables of that system:

$$x = y(x_1, x_2, x_3),$$

$$y = y(x_1, x_2, x_3),$$

$$z = z(x_1, x_2, x_3).$$

Then

$$ds_i = \left[\left(\frac{\partial x}{\partial x_i}\right)^2 + \left(\frac{\partial y}{\partial x_i}\right)^2 + \left(\frac{\partial z}{\partial x_i}\right)^2\right]^{1/2} dx_i, \quad i = 1, 2, 3. \tag{A.25}$$

Hence,

$$h_i = \left[\left(\frac{\partial x}{\partial x_i}\right)^2 + \left(\frac{\partial y}{\partial x_i}\right)^2 + \left(\frac{\partial z}{\partial x_i}\right)^2\right]^{1/2}. \tag{A.26}$$

For instance, in an elliptical coordinate system,

$$x = c\cosh u \cos v, \tag{A.27}$$

$$y = c\sinh u \sin v. \tag{A.28}$$

If (x_1, x_2, x_3) represent (u, v, z), then by applying (26), we have

$$h_1 = h_2 = c(\sinh^2 u \cos^2 v + \cosh^2 u \sin^2 v)^{1/2} = c(\cosh^2 u - \cos^2 v)^{1/2}, \tag{A.29}$$

$$h_3 = 1. \tag{A.30}$$

In general, for any orthogonal curvilinear coordinate system,

$$\nabla\psi = \sum_{i=1}^{3} \frac{1}{h_i}\frac{\partial\psi}{\partial x_i}\hat{x}_i, \tag{A.31}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{\Delta}\sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left(\frac{\Delta A_i}{h_i}\right), \quad \Delta = h_1 h_2 h_3, \tag{A.32}$$

$$\nabla \times \mathbf{A} = \frac{1}{\Delta}\left|\begin{pmatrix} h_1\hat{x}_1 & h_2\hat{x}_2 & h_3\hat{x}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3, \end{pmatrix}\right|, \tag{A.33}$$

$$\nabla^2\psi = \frac{1}{\Delta}\sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left(\frac{\Delta}{h_i^2}\frac{\partial\psi}{\partial x_i}\right). \tag{A.34}$$

A.3 Useful Integral Identities

In the following formulas, V is a volume bounded by a closed surface S. The unit vector $\hat{n}$ is normal to S and points outward.

(a) Gradient Identity:

$$\oint_V \nabla\phi \, dV = \oint_S \phi\hat{n} \, dS. \tag{A.35}$$

(b) Gauss' Divergence Theorem:

$$\oint_V \nabla \cdot \mathbf{A}\, dV = \oint_S \mathbf{A} \cdot \hat{n}\, dS. \tag{A.36}$$

(c) Vector Stokes' Theorem:

$$\oint_V \nabla \times \mathbf{A}\, dV = \oint_S \hat{n} \times \mathbf{A}\, dS. \tag{A.37}$$

(d) First Form of Green's Theorem:

$$\oint_V [\phi_1 \nabla^2 \phi_2 + \nabla\phi_1 \cdot \nabla\phi_2]\, dV = \oint_S \hat{n} \cdot \phi_1 \nabla\phi_2\, dS. \tag{A.38}$$

(e) Second Form of Green's Theorem:

$$\oint_V [\phi_1 \nabla^2 \phi_2 - \phi_2 \nabla^2 \phi_1]\, dV = \oint_S \hat{n} \cdot (\phi_1 \nabla\phi_2 - \phi_2 \nabla\phi_1)\, dS. \tag{A.39}$$

(f) Vector Green's Theorem:

$$\oint_V [\mathbf{P} \cdot \nabla \times \nabla \times \mathbf{Q} - \mathbf{Q} \cdot \nabla \times \nabla \times \mathbf{P}]\, dV$$
$$= \oint_S [\mathbf{Q} \times \nabla \times \mathbf{P} - \mathbf{P} \times \nabla \times \mathbf{Q}] \cdot \hat{n}\, dS. \tag{A.40}$$

The above may all be proved from Gauss' divergence theorem.

(g) Stokes' Theorem:

If S is an unclosed surface bounded by a contour C, then

$$\int_S (\nabla \times \mathbf{A}) \cdot \hat{n}\, dS = \oint_C \mathbf{A} \cdot d\mathbf{l}, \tag{A.41}$$

$$\int_S \hat{n} \times \nabla\phi\, dS = \oint_C \phi\, d\mathbf{l}. \tag{A.42}$$

(h) Gauss' Theorem in Two Dimensions:

$$\int_S (\nabla \cdot \mathbf{A})\, dS = \oint_C \mathbf{A} \cdot \hat{n}\, dl, \tag{A.43}$$

The above identities for tensors and dyads can also be readily established (see Appendix B).

A.4 Integral Transforms

(a) Fourier:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dy\, e^{ixy} \tilde{f}(y), \tag{A.44}$$

$$\tilde{f}(y) = \int_{-\infty}^{\infty} dx\, e^{-ixy} f(x), \tag{A.45}$$

$$\delta(x - x') = \frac{1}{2\pi} \int_{-\infty}^{\infty} dy\, e^{i(x-x')y}. \tag{A.46}$$

(b) Cylindrical Hankel:

$$f(\rho) = \int_0^{\infty} d\lambda\, \lambda J_n(\lambda\rho) \tilde{f}(\lambda), \tag{A.47}$$

$$\tilde{f}(\lambda) = \int_0^{\infty} d\rho\, \rho J_n(\lambda\rho) f(\rho), \tag{A.48}$$

$$\frac{\delta(\rho - \rho')}{\rho} = \int_0^{\infty} d\lambda\, \lambda J_n(\lambda\rho) J_n(\lambda\rho'), \tag{A.49}$$

where $J_n(x)$ is a cylindrical Bessel function of n-th order.

(c) Spherical Hankel:

$$f(r) = \left(\frac{2}{\pi}\right)^{1/2} \int_0^{\infty} d\lambda\, \lambda^2 j_n(\lambda r) \tilde{f}(\lambda), \tag{A.50}$$

$$\tilde{f}(\lambda) = \left(\frac{2}{\pi}\right)^{1/2} \int_0^{\infty} dr\, r^2 j_n(\lambda r) f(r), \tag{A.51}$$

$$\frac{\delta(r-r')}{r^2} = \frac{2}{\pi}\int_0^{\infty} d\lambda\, \lambda^2 j_n(\lambda r) j_n(\lambda r'), \tag{A.52}$$

where $j_n(x)$ is a spherical Bessel function of n-th order.

(d) Hilbert:

$$g(t) = \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} d\tau\, \frac{f(\tau)}{\tau - t}, \tag{A.53}$$

$$f(\tau) = -\frac{1}{\pi} P.V. \int_{-\infty}^{\infty} dt\, \frac{g(t)}{t - \tau}, \tag{A.54}$$

$$\delta(t-t') = -\frac{1}{\pi^2} P.V. \int_{-\infty}^{\infty} d\tau\, \frac{1}{\tau - t} P.V. \frac{1}{t' - \tau}, \tag{A.55}$$

where $P.V.\frac{1}{x}$ is regarded as a generalized function (see Appendix C).

(e) Radon:

A Radon transform in an n-dimensional space is defined as

$$\tilde{f}(p,\hat{\xi}) = \int_{-\infty}^{\infty} f(\mathbf{x})\, \delta\left(p - \hat{\xi}\cdot\mathbf{x}\right) d\mathbf{x}, \tag{A.56}$$

where $\mathbf{x}$ is a position vector in an n-dimensional space, $\hat{\xi}$ is a unit vector, and $d\mathbf{x}$ implies a volume integral in an n-dimensional space. The inverse Radon transform is different in even and odd dimensions. In even dimensions, it is

$$f(\mathbf{x}) = \frac{1}{(2\pi i)^n} \int_{|\xi|=1} d\xi \int_{-\infty}^{\infty} dp\, \frac{1}{p - \hat{\xi}\cdot\mathbf{x}} \left(\frac{\partial}{\partial p}\right)^{n-1} \tilde{f}(p,\hat{\xi}), \tag{A.57}$$

where the $d\xi$ integral implies integrating over all angles of $\hat{\xi}$, i.e., $d\xi$ is an elemental area on the surface of a unit sphere. In odd dimensions,

$$f(\mathbf{x}) = \frac{1}{2}\frac{1}{(2\pi i)^{n-1}} \int_{|\xi|=1} d\xi \left(\frac{\partial}{\partial p}\right)^{n-1} \tilde{f}(p,\hat{\xi})\Big|_{p=\xi\cdot\mathbf{x}}. \tag{A.58}$$