34 Line impedance, generalized reflection coefficient, Smith Chart

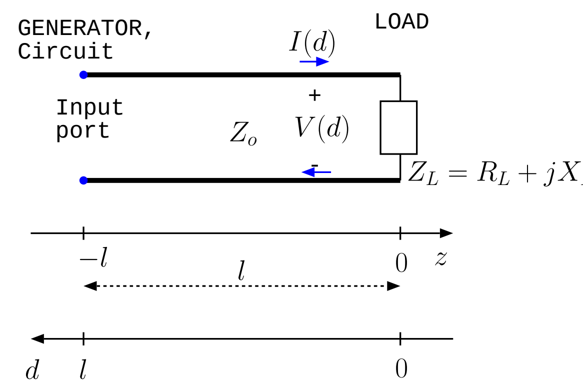
- Consider a TL of an arbitrary length l terminated by an arbitrary load

$$Z_L = R_L + jX_L.$$

as depicted in the margin.

Voltage and current phasors are known to vary on the line as

$$V(d) = V^+e^{j\beta d} + V^-e^{-j\beta d} \quad \text{and} \quad I(d) = \frac{V^+e^{j\beta d} - V^-e^{-j\beta d}}{Z_o}.$$



In this lecture we will develop the general analysis tools needed to determine the unknowns of these phasors, namely V^+ and V^-, in terms of source circuit specifications.

- Our analysis starts at the load end of the TL where $V(0)$ and $I(0)$ stand for the load voltage and current, obeying Ohm's law

$$V(0) = Z_L I(0).$$

Hence, using $V(0)$ and $I(0)$ from above, we have

$$V^+ + V^- = Z_L \frac{V^+ - V^-}{Z_o} \quad \Rightarrow \quad V^- = \frac{Z_L - Z_o}{Z_L + Z_o} V^+.$$

– Define a **load reflection coefficient**

$$\Gamma_L \equiv \frac{Z_L - Z_o}{Z_L + Z_o}$$

and re-write the voltage and current phasors as

$$V(d) = V^+ e^{j\beta d}[1 + \Gamma_L e^{-j2\beta d}] \quad \text{and} \quad I(d) = \frac{V^+ e^{j\beta d}[1 - \Gamma_L e^{-j2\beta d}]}{Z_o}.$$

"Load reflection coefficient" is a well justified name for Γ_L since the forward traveling wave with phasor $V^+ e^{j\beta d}$ gets reflected from the load.

– Define a **generalized reflection coefficient**

$$\Gamma(d) \equiv \Gamma_L e^{-j2\beta d}$$

and re-write the voltage and current phasors as

$$V(d) = V^+ e^{j\beta d}[1 + \Gamma(d)] \quad \text{and} \quad I(d) = \frac{V^+ e^{j\beta d}[1 - \Gamma(d)]}{Z_o}.$$

The term "generalized reflection coefficient" is also well justified even if there is no reflection taking place at arbitrary d — the reason is, if the line were cut at location d and the stub with the load were replaced by a lumped load having a reflection coefficient equal to $\Gamma(d)$, then there would be no modification of the voltage and current variations on the line towards the generator.

– **Line impedance** is then defined as

$$Z(d) = \frac{V(d)}{I(d)} = Z_o \frac{1 + \Gamma(d)}{1 - \Gamma(d)}$$

for all values of d on the line extending from the load point $d = 0$ all the way to the input port at $d = l$.

With the dependence on d of $Z(d)$ as well as $\Gamma(d)$ tacitly implied, we can re-write this important relation and its inverse as

$$\frac{Z}{Z_o} = \frac{1 + \Gamma}{1 - \Gamma} \quad \Leftrightarrow \quad \Gamma = \frac{Z - Z_o}{Z + Z_o}.$$

Each location d on the line has an impedance Z and a reflection coefficient Γ linked by these equations.

Properties of $Z(d) = R(d) + jX(d)$ and $\Gamma(d) = \Gamma_L e^{-j2\beta d}$ linked by the relations

$$\frac{Z}{Z_o} = \frac{1+\Gamma}{1-\Gamma} \quad \Leftrightarrow \quad \Gamma = \frac{Z - Z_o}{Z + Z_o}:$$

1. **For real valued Z_o and $R(d) \geq 0$, $|\Gamma(d)| \leq 1$:**

   Verification:

   $$|\Gamma| = \frac{|Z - Z_o|}{|Z + Z_o|} = \frac{|(R - Z_o) + jX|}{|(R + Z_o) + jX|} = \frac{\sqrt{(R - Z_o)^2 + X^2}}{\sqrt{(R + Z_o)^2 + X^2}}.$$

   Since with $R \geq 0$

   $$\sqrt{(R - Z_o)^2 + X^2} \leq \sqrt{(R + Z_o)^2 + X^2} \quad \Rightarrow \quad |\Gamma| \leq 1.$$

2. Since

   $$|\Gamma| = |\Gamma_L| \quad \text{and} \quad \angle\Gamma(d) = \angle\Gamma_L - 2\beta d$$

   property (1) implies that $\Gamma(d)$ is a complex number which is constrained to be **on or within the unit-circle on the complex plane**.

3. Relationships

   $$\frac{Z}{Z_o} = \frac{1+\Gamma}{1-\Gamma} \quad \Leftrightarrow \quad \Gamma = \frac{Z - Z_o}{Z + Z_o}$$

   between Γ and Z are known as **bilinear transformations** — here the term *bilinear* refers to the numerator *as well as* the denominator of these transformations being *linear* in the variable being transformed (from right to left).

Bilinear (or Möbius) transformations are known to have the general property of mapping **straight lines** into **circles** on the complex number plane.

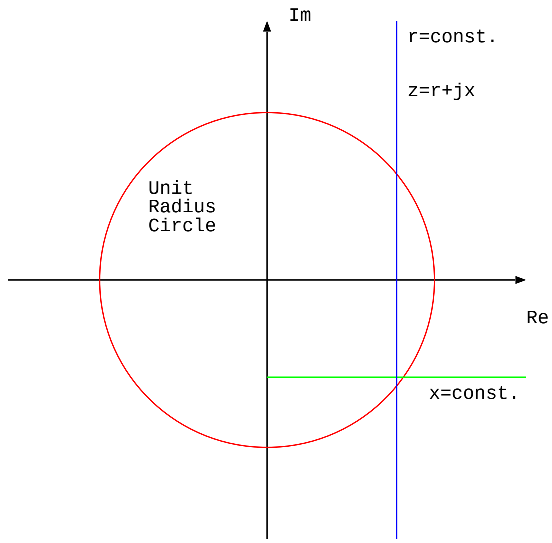
- Bilinear transformations between

$$\Gamma \equiv \Gamma_r + j\Gamma_i \equiv (\Gamma_r, \Gamma_i)$$

and

$$\frac{Z}{Z_o} \equiv z \equiv r + jx,$$

known as **normalized impedance,** lead to an ingenious graphical aid known as the **Smith Chart**.



  - On a Smith Chart (SC), straight lines on the right hand side of the complex number plane (see margin), represented by

$$r = \text{const.} \quad \text{and} \quad x = \text{const.},$$

are mapped onto circular loci of

$$(\Gamma_r, \Gamma_i) = \Gamma = \frac{Z - Z_o}{Z + Z_o} = \frac{z-1}{z+1}$$

occupying the region of the plane bordered by the unit circle.

Circles corresponding to $z = \text{const.} + jx$ and $z = r + j\text{const.}$constitute a **gridding** of the unit circle and its interior. By means of this **grid**, the normalized impedance z corresponding to every possible Γ can be directly read off the SC.

- SC can be constructed by first noting that

$$\Gamma = \frac{z-1}{z+1} = \frac{r+jx-1}{r+jx+1} = \frac{[(r-1)+jx][(r+1)-jx]}{(r+1)^2+x^2} = \frac{(r^2+x^2-1)+j2x}{(r+1)^2+x^2} \equiv \Gamma_r + j\Gamma_i;$$

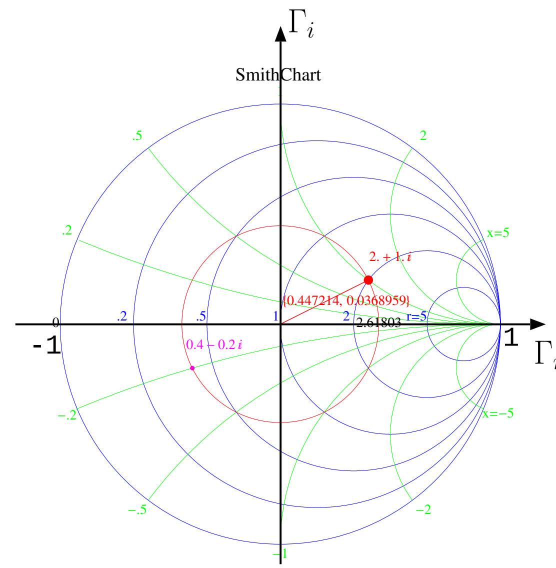
thus

$$\Gamma_r = \frac{(r^2+x^2-1)}{(r+1)^2+x^2} \text{ and } \Gamma_i = \frac{2x}{(r+1)^2+x^2},$$

and by direct substitution we can verify the following equations

$$(\Gamma_r - \frac{r}{r+1})^2 + \Gamma_i^2 = (\frac{1}{r+1})^2 \text{ and } (\Gamma_r - 1)^2 + (\Gamma_i - \frac{1}{x})^2 = (\frac{1}{x})^2$$

describing r and x dependent circles, respectively, on complex plane constituting the grid lines of the SC.



Smith Chart **is the unit circle and its interior on the complex number plane corresponding to the *generalized reflection coefficient* Γ. The gridding allows direct identification of the bilinear transform of Γ, namely the *normalized line impedance* z.**

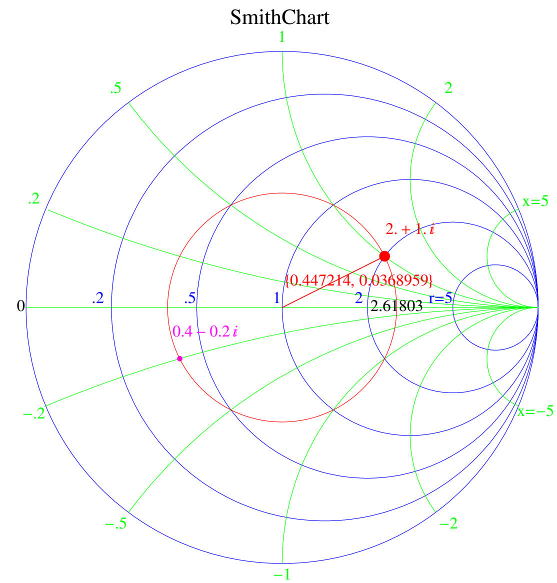
- Typical SC usage:

  1. Locate and mark $z(0)$ — normalized load impedance — on the SC, which places you at a distance $|\Gamma(0)| = |\Gamma_L|$ from the origin of the complex plane (and the SC), at an angle of $\theta = \angle\Gamma(0)$.
  2. Draw a constant $|\Gamma| = |\Gamma_L|$ circle with a compass going through point $z(0)$ on the SC (the read circle in the margin). Rotate clockwise on the circle by an angle of

$$2\beta d = \frac{4\pi}{\lambda} d \,\text{rad} = \frac{d}{\lambda/2} 360°$$

  to land on $z(d)$ that can be read off using the SC gridding.

– Rotation by an angle of $2\beta d$ amounts to rotation by full circle for $d = \frac{\lambda}{2}$, rotation by half circle for $d = \frac{\lambda}{4}$, rotation by quarter circle for $d = \frac{\lambda}{8}$, etc.



3. Also,

$$y(d) \equiv \frac{1}{z(d)}$$

which is the **normalized line admittance** is located on the SC on the constant $|\Gamma| = |\Gamma_L|$ circle across the point corresponding to $z(d)$.

Verification: Since

$$z = \frac{1+\Gamma}{1-\Gamma} \quad \Rightarrow \quad y = \frac{1}{z} = \frac{1-\Gamma}{1+\Gamma} = \frac{1+(-\Gamma)}{1-(-\Gamma)};$$

hence whereas z is the transform of Γ, y is the transform of $-\Gamma$, having the same magnitude as Γ but an angle off by $\pm 180°$.

– Therefore, **"reflect" on the SC across the origin** to jump from $z(d)$ to $y(d)$ if you need the value of the normalized admittance.

Our first SC example is given next.

Example 1: A transmission line is terminated by an inductive load of

$$Z_L = 50 + j100\,\Omega.$$

Determine the input impedance $Z_{in} = Z(l)$ of the line at a distance

$$d = l = \frac{\lambda}{8}$$

if the characteristic impedance of the line is $Z_o = 50\,\Omega$. Also determine the normalized input admittance $y(l)$.

Solution: The normalized load impedance is

$$z(0) = \frac{Z_L}{Z_o} = \frac{50 + j100}{50} = 1 + j2.$$

Enter $z(0)$ on the SC and then rotate clockwise by $\frac{\lambda}{8} \Leftrightarrow$(quarter circle) to obtain the normalized input impedance

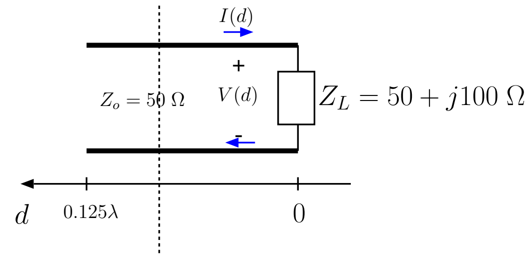
$$z(l) = 1 - j2,$$

and the normalized input admittance

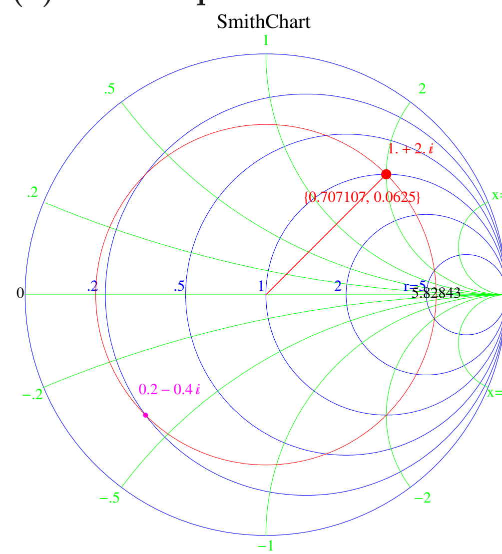
$$y(l) = 0.2 + j0.4$$

right across $z(l)$. The input impedance is

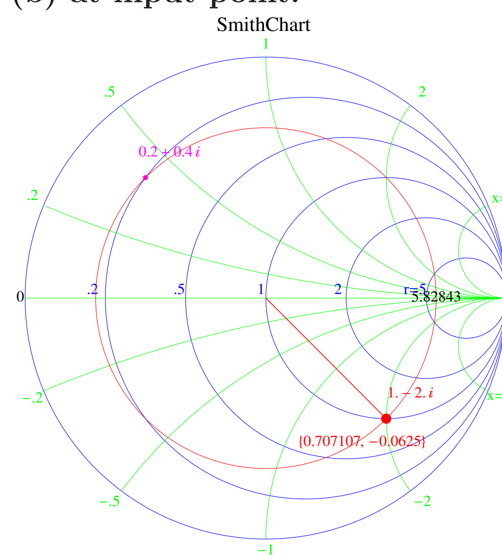
$$Z_{in} = Z_o z(l) = 50(1 - j2) = 50 - j100\,\Omega.$$



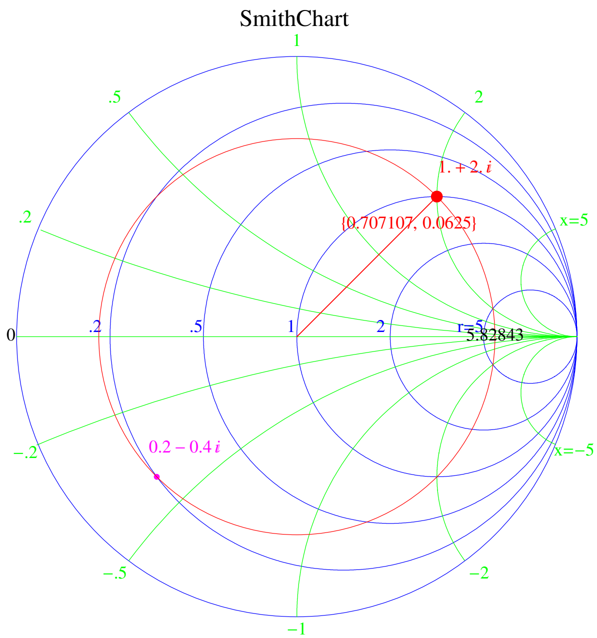
(a) At load point:



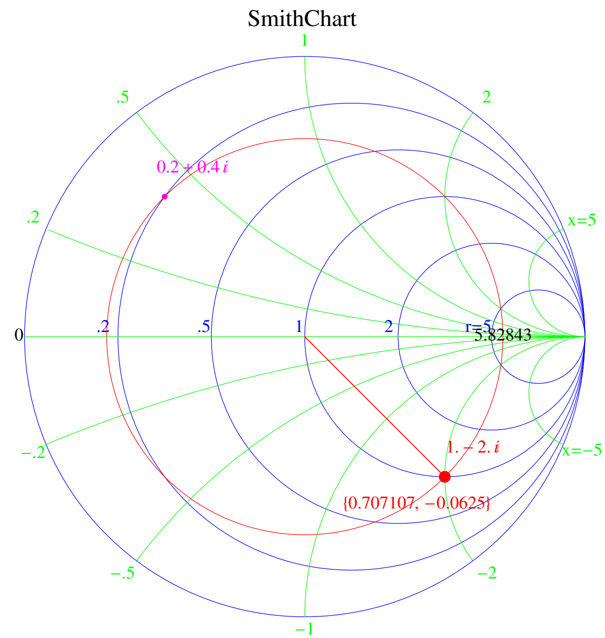
(b) at input point:



Blow up of the SC's used in Example 1:



(a) At load point



(b) at input point

- A SmithChartTool linked from the class calendar (a javascript utility that requires a Safari or Firefox browser to work properly) marks and prints $z(d)$ in **red** and $y(d)$ in **magenta** across from $z(d)$ on the constant-$|\Gamma_L|$ circle (shown in red) as in the above examples. Also
  - printed in **black** is the real valued normalized impedance $z(d_{max})$ discussed in the upcoming lectures (also known as VSWR).
  - also printed in **red** is $|\Gamma_L|\angle\Gamma(d)$ where the second entry is expressed in terms of an equivalent $\frac{d}{\lambda}$ such that $\frac{d}{\lambda} = 0.5$ corresponds to an angle of 360°. This way of referring to $\angle\Gamma(d)$ will be convenient in many SC applications that we will see.