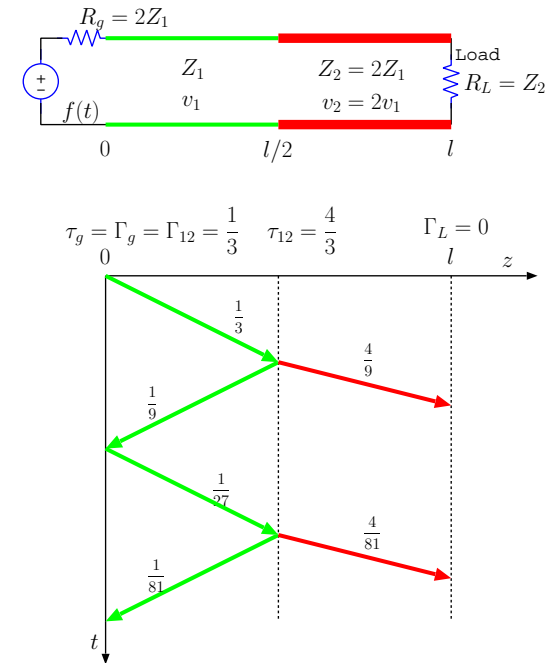


30 Multi-line circuits

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- In this lecture we will extend the bounce diagram technique to solve distributed circuit problems involving multiple transmission lines.
- One example of such a circuit is shown in the margin where two distinct TL's of equal lengths have been joined directly at a distance $\frac{l}{2}$ away from the generator.
 - The impulse response of the system can be found by first constructing the bounce diagram for the TL system as shown in the margin.
 - In this bounce diagram, $z = \frac{l}{2}$ happens to be the location of additional reflections as well as transmissions because of the sudden change of Z_o from Z_1 to $Z_2 = 2Z_2$.



These reflections and transmissions between line j and k — transmission from j to k , and reflection from k back to j — can be computed with **reflection coefficient**

$$\Gamma_{jk} = \frac{Z_k - Z_j}{Z_k + Z_j}$$

and **transmission coefficient**

$$\tau_{jk} = 1 + \Gamma_{jk}$$

that ensure the voltage and current continuity at the junction

- Z_j is the characteristic impedance of the line of the incident pulse, while
- Z_k is the impedance of the cascaded line into which the transmitted pulse is injected.

Verification:

- Let

$$V_j^+(1 + \Gamma_{jk}) \quad \text{and} \quad V_j^+(1 - \Gamma_{jk})/Z_j$$

denote the total voltage and current on line Z_j expressed in terms of incident voltage wave V^+ (of d'Alembert type), and

- let

$$V_j^+ \tau_{jk} \quad \text{and} \quad V_j^+ \tau_{jk}/Z_k$$

denote the voltage and current on line Z_k adjacent to line Z_j .

This notation identifies Γ_{jk} and τ_{jk} as reflection and transmission coefficients at the junction.

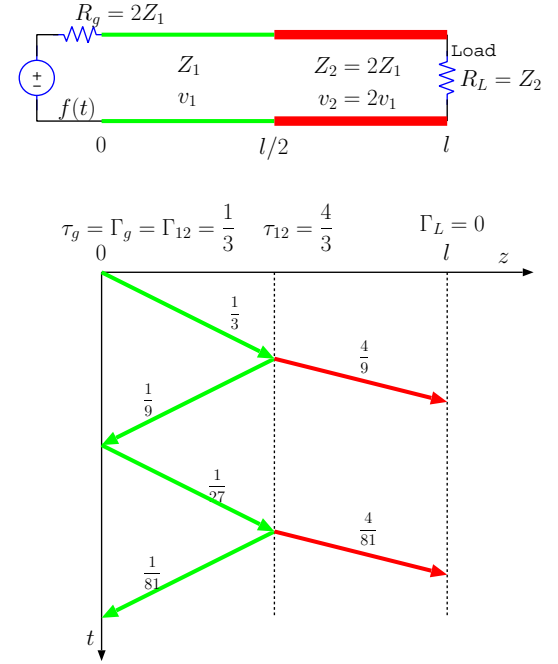
- Taking

$$V_j^+(1 + \Gamma_{jk}) = V_j^+ \tau_{jk}$$

and

$$V_j^+(1 - \Gamma_{jk})/Z_j = V_j^+ \tau_{jk}/Z_k$$

in order to enforce voltage and current continuity, we can solve for Γ_{jk} and τ_{jk} expressions stated above.



Example 1: In the circuit shown in the margin with two TL segments, line 2 has twice the characteristic impedance and propagation velocity of line 1, i.e.,

$$Z_2 = 2Z_1 \quad \text{and} \quad v_2 = 2v_1.$$

Determine \mathcal{L}_2 and \mathcal{C}_2 in terms of \mathcal{L}_1 and \mathcal{C}_1 .

Solution: We have

$$Z_2 = 2Z_1 \Rightarrow \frac{\mathcal{L}_2}{\mathcal{C}_2} = 4 \frac{\mathcal{L}_1}{\mathcal{C}_1}$$

and

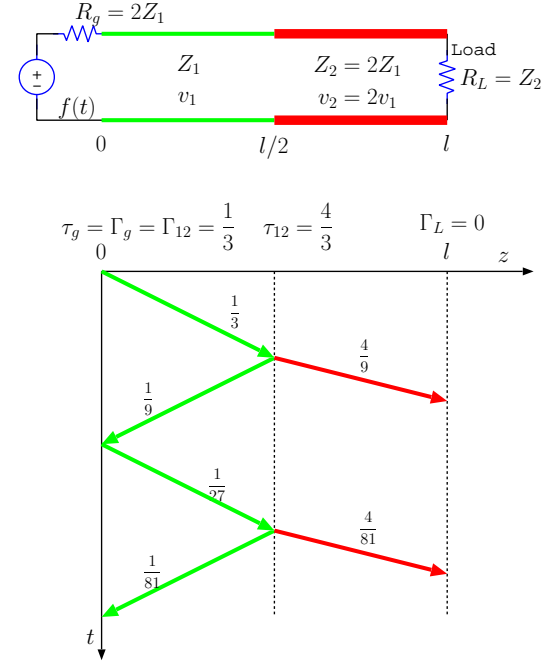
$$v_2 = 2v_1 \Rightarrow \frac{1}{\mathcal{L}_2 \mathcal{C}_2} = 4 \frac{1}{\mathcal{L}_1 \mathcal{C}_1}.$$

The product of the two equations gives

$$\frac{1}{\mathcal{C}_2^2} = 16 \frac{1}{\mathcal{C}_1^2} \Rightarrow \mathcal{C}_2 = \frac{1}{4} \mathcal{C}_1,$$

while their ratio leads to

$$\mathcal{L}_2 = \mathcal{L}_1.$$



Example 2: In the circuit of Example 1, determine $V(z, t)$ and $I(z, t)$ if

$$f(t) = \sin(2\pi t)u(t), \quad t \text{ in } \mu s,$$

and $l = 2400$ m, $v_1 = 150$ m/ μs , and $Z_1 = 25 \Omega$.

Solution: From the bounce diagram we infer the following impulse-response for the voltage variable:

$$V(z, t) = \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^{2n} \left[\delta\left(t - \frac{z}{v_1} - n\frac{l}{v_1}\right) + \frac{1}{3} \delta\left(t + \frac{z}{v_1} - (n+1)\frac{l}{v_1}\right) \right]$$

for $z < \frac{l}{2}$, and

$$V(z, t) = \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^{2n} \frac{4}{3} \delta\left(t - \frac{z}{v_2} - (4n+1)\frac{l/2}{v_2}\right)$$

for $\frac{l}{2} < z < l$. The impulse response for the current is

$$I(z, t) = \frac{1}{3Z_1} \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^{2n} \left[\delta\left(t - \frac{z}{v_1} - n\frac{l}{v_1}\right) - \frac{1}{3} \delta\left(t + \frac{z}{v_1} - (n+1)\frac{l}{v_1}\right) \right]$$

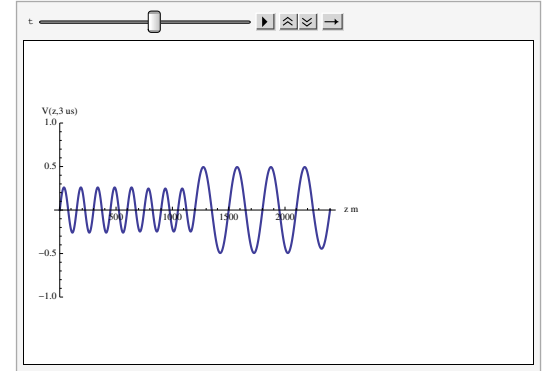
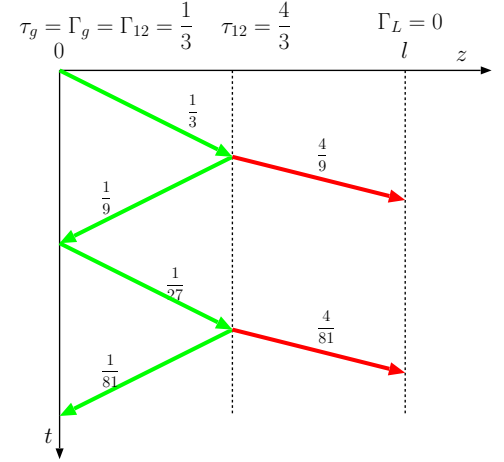
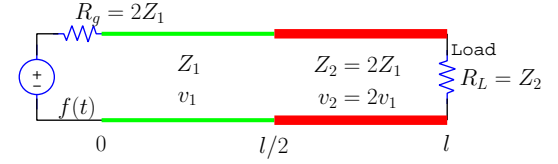
for $z < \frac{l}{2}$, and

$$I(z, t) = \frac{1}{3Z_2} \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^{2n} \frac{4}{3} \delta\left(t - \frac{z}{v_2} - (4n+1)\frac{l/2}{v_2}\right)$$

for $\frac{l}{2} < z < l$. Using

$$\frac{l}{v_1} = \frac{2400}{150} = 16 \mu s$$

and replacing $\delta(t)$ with $f(t) = \sin(2\pi t)u(t)$ the plot depicted in the margin was obtained.



Example 3: Two TL's with characteristic impedances Z_1 and Z_2 are joined at a junction that also includes a “shunt” resistance R as shown in the diagram in the margin. Determine the reflection coefficient Γ_{12} and transmission coefficient τ_{12} at the junction.

Solution: Consider a voltage wave

$$V^+(t - \frac{z}{v_1})$$

coming from the left producing reflected and transmitted waves

$$V^-(t + \frac{z}{v_1}) \quad \text{and} \quad V^{++}(t - \frac{z}{v_2})$$

on lines 1 and 2 traveling to the left and right, respectively, on two sides of the junction. Using an abbreviated notation, KVL and KCL applied at the junction can be expressed as

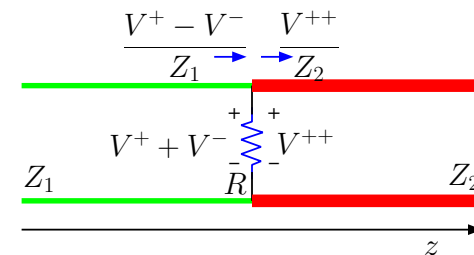
$$V^+ + V^- = V^{++} \quad \text{and} \quad \frac{V^+}{Z_1} - \frac{V^-}{Z_1} = \frac{V^{++}}{R} + \frac{V^{++}}{Z_2},$$

where in the KCL equation the first term on the right is the current flowing down the resistor R , and the second term is the TL current on line 2 (as marked in the circuit diagrams in the margin). The equations can be rearranged as

$$\begin{aligned} V^+ + V^- &= V^{++} \\ V^+ - V^- &= \frac{Z_1}{Z_{eq}} V^{++}, \end{aligned}$$

where

$$Z_{eq} \equiv \frac{RZ_2}{R + Z_2}$$



is the parallel combination of R and Z_2 . Solving these equations, we find that

$$\Gamma_{12} \equiv \frac{V^-}{V^+} = \frac{Z_{eq} - Z_1}{Z_{eq} + Z_1}$$

and

$$\tau_{12} = \frac{V^{++}}{V^+} = \frac{2Z_{eq}}{Z_{eq} + Z_1}.$$

By, symmetry, the coefficients

$$\Gamma_{21} = \frac{Z_{eq} - Z_2}{Z_{eq} + Z_2}$$

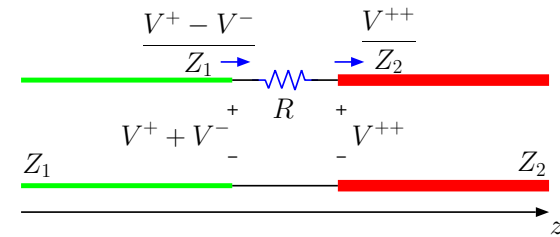
and

$$\tau_{21} = \frac{2Z_{eq}}{Z_{eq} + Z_2}$$

would describe reflection and transmission when a wave is incident from right provided that

$$Z_{eq} \equiv \frac{RZ_1}{R + Z_1}$$

is used.



Hint: in this ckt Γ_{12} has the usual form in terms of $Z_{eq} \equiv R + Z_2$. For τ_{12} , we need $1 + \Gamma_{12}$ multiplied by a voltage division factor $Z_2/(R + Z_2)$.

Exercise: Two TL's with characteristic impedances Z_1 and Z_2 are joined at a junction that also includes a series resistance R as shown in the margin. Determine the reflection coefficient Γ_{12} and transmission coefficient τ_{12} at the junction.