28 Distributed circuits and bounce diagrams

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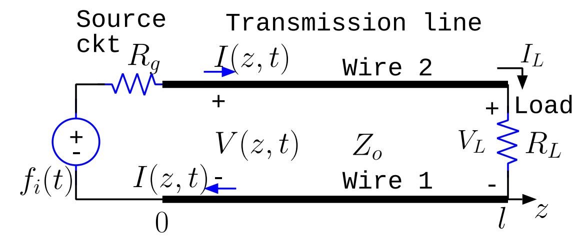
Last lecture we learned that voltage and current variations on TL's are governed by telegrapher's equations and their d'Alembert solutions — the latter can be expressed as

$$V(z,t) = f(t - \frac{z}{v}) + g(t + \frac{z}{v}) \text{ and } I(z,t) = \frac{f(t - \frac{z}{v})}{Z_o} - \frac{g(t + \frac{z}{v})}{Z_o}$$

in terms of

$$v = \frac{1}{\sqrt{\mathcal{LC}}} \text{ and } Z_o = \sqrt{\frac{\mathcal{L}}{\mathcal{C}}}$$

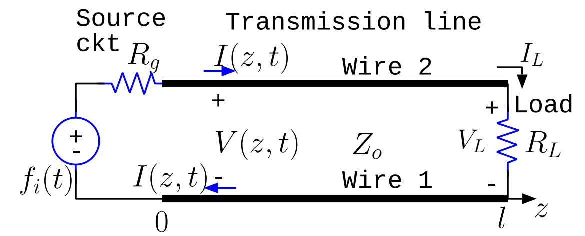
and functions $f(t)$ and $g(t)$ corresponding to signal waveforms propagated in $+z$ and $-z$ directions, respectively.



- In this lecture we will learn how to solve **distributed circuit problems** containing TL segments *and* two terminal elements such as resistors and voltage (or current) sources. In solving the problems, we will apply the usual rules of lumped circuit analysis at element terminals and treat the TL's in terms of d'Alembert solutions above.

- Consider a TL with a characteristic impedance Z_o extending from $z = 0$ to $z = l$, where a two-terminal *source* circuit (e.g., a receiving antenna) modeled by a Thevenin equivalent with voltage $f_i(t)$ and resistance R_g is connected between the TL terminals at $z = 0$ and a *load* (e.g., a receiver circuit) modeled by a resistance R_L terminates the line at $z = l$ (see margin).

– We want to determine voltage and current signals $V(z,t)$ and $I(z,t)$ on the TL and the load R_L for time $t > 0$ in terms of source signal $f_i(t)$ assuming that $f_i(t) = 0$ for $t < 0$.



- Using the d'Alembert solutions $V(z,t)$ and $I(z,t)$ from above at $z = l$, we have

$$\frac{V(\ell,t)}{I(\ell,t)} = \frac{f(t-\frac{\ell}{v}) + g(t+\frac{\ell}{v})}{\frac{f(t-\frac{\ell}{v})}{Z_o} - \frac{g(t+\frac{\ell}{v})}{Z_o}} = Z_o \frac{f(t-\frac{\ell}{v}) + g(t+\frac{\ell}{v})}{f(t-\frac{\ell}{v}) - g(t+\frac{\ell}{v})} = \frac{V_L}{I_L} = R_L,$$

from which we obtain

$$g(t+\frac{l}{v}) = \underbrace{\frac{R_L - Z_o}{R_L + Z_o}}_{\Gamma_L} f(t-\frac{l}{v}) \quad \Rightarrow \quad g(t) = \Gamma_L f(t - \frac{2l}{v})$$

where

$$\Gamma_L = \frac{R_L - Z_o}{R_L + Z_o}$$

is the **load reflection coefficient** in the TL circuit. We can re-write the d'Alembert solution for $V(z,t)$ and $I(z,t)$ in terms of only $f(t)$ as

$$V(z,t) = f(t-\frac{z}{v}) + \Gamma_L f(t+\frac{z}{v}-\frac{2l}{v}) \quad \text{and} \quad I(z,t) = \frac{f(t-\frac{z}{v})}{Z_o} - \frac{\Gamma_L f(t+\frac{z}{v}-\frac{2l}{v})}{Z_o}.$$

- Assuming that $f_i(t) = 0 = f(t)$ for $t < 0$, we can relate $f(t)$ to $f_i(t)$ in $t > 0$ interval using the KVL equation at $z = 0$ that states

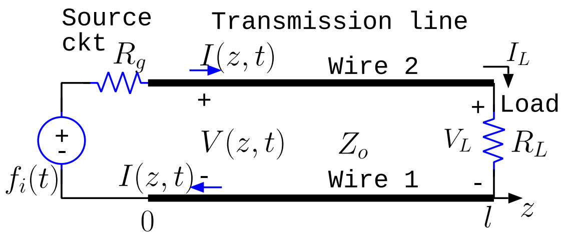
$$f_i(t) = R_g I(0,t) + V(0,t),$$

which is, using $V(z,t)$ and $I(z,t)$ at $z=0$,

$$f_i(t) = R_g \underbrace{\left(\frac{f(t)}{Z_o} - \frac{\Gamma_L f(t-\frac{2l}{v})}{Z_o}\right)}_{I(0,t)} + \underbrace{f(t) + \Gamma_L f(t-\frac{2l}{v})}_{V(0,t)}.$$

Now, since $f(t-\frac{2l}{v}) = 0$ for $t - \frac{2l}{v} < 0$, we find out that for the *epoch* (or *time interval*) $0 < t < \frac{2l}{v}$,

$$f_i(t) = R_g \frac{f(t)}{Z_o} + f(t) \quad \Rightarrow \quad f(t) = \underbrace{\frac{Z_o}{R_g + Z_o}}_{\tau_g} f_i(t)$$



where

$$\tau_g = \frac{Z_o}{R_g + Z_o}$$

is the **injection coefficient** of the TL circuit[1].

- Thus, for the epoch $0 < t < \frac{2l}{v}$, we have the voltage and current solutions

$$V(z,t) = \tau_g f_i(t - \frac{z}{v}) + \Gamma_L \tau_g f_i(t + \frac{z}{v} - \frac{2l}{v}) \quad \text{and} \quad I(z,t) = \frac{\tau_g f_i(t-\frac{z}{v})}{Z_o} - \frac{\Gamma_L \tau_g f_i(t + \frac{z}{v} - \frac{2l}{v})}{Z_o}$$

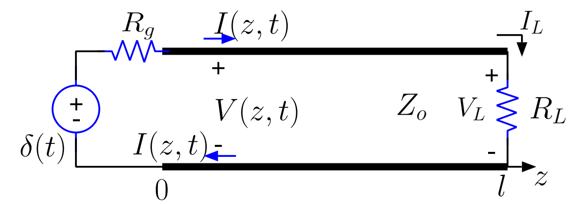
on the line.

[1]Note how $f(t)$ appears to be related to $f_i(t)$ according to a voltage division rule with Z_o representing the resistance across which voltage $f(t)$ is measured.

– So far $f_i(t)$ function is arbitrary and the above results would also be valid for $f_i(t) = \delta(t)$, Dirac's impulse, in which case

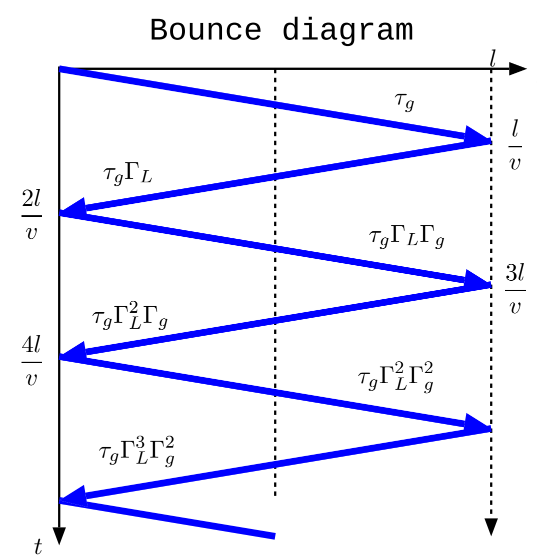
$$V(z,t) = \tau_g \delta(t - \frac{z}{v}) + \Gamma_L \tau_g \delta(t + \frac{z}{v} - \frac{2l}{v}) \text{ and } I(z,t) = \frac{\tau_g \delta(t - \frac{z}{v})}{Z_o} - \frac{\Gamma_L \tau_g \delta(t + \frac{z}{v} - \frac{2l}{v})}{Z_o}$$

would be the voltage and current **impulse response functions** of the TL circuit for the $0 < t < \frac{2l}{v}$ epoch.



- To extend the impulse response functions above to the "next epoch" $\frac{2l}{v} < t < \frac{4l}{v}$, we note that at $z = 0$ the KVL equation with $f_i(t) = \delta(t)$ reads as

$$\delta(t) = R_g \underbrace{(\frac{f(t)}{Z_o} - \frac{\Gamma_L f(t - \frac{2l}{v})}{Z_o})}_{I(0,t)} + \underbrace{f(t) + \Gamma_L f(t - \frac{2l}{v})}_{V(0,t)}.$$



which can be re-arranged as

$$\delta(t) = (1 + \frac{R_g}{Z_o}) f(t) + (1 - \frac{R_g}{Z_o}) \Gamma_L \underbrace{f(t - \frac{2l}{v})}_{\tau_g \delta(t - \frac{2l}{v})},$$

where for $f(t - \frac{2l}{v})$ we used a delayed copy of $f(t) = \tau_g f_i(t)$ solution for $f(t)$ from the *previous epoch* in view of the time delay $\frac{2l}{v}$ contained within $f(t - \frac{2l}{v})$.

– Hence, solving this for $f(t)$, we find, for *this* epoch,

$$f(t) = \tau_g \delta(t) + \underbrace{\frac{R_g - Z_o}{R_g + Z_o}}_{\Gamma_g} \Gamma_L \tau_g \delta(t - \frac{2l}{v}),$$

where

$$\Gamma_g = \frac{R_g - Z_o}{R_g + Z_o}$$

is the **source reflection coefficient** of the TL circuit.

– Substituting $f(t)$ for the epoch $\frac{2l}{v} < t < \frac{4l}{v}$ within voltage and current formulae

$$V(z,t) = f(t - \frac{z}{v}) + \Gamma_L f(t + \frac{z}{v} - \frac{2l}{v}) \text{ and } I(z,t) = \frac{f(t-\frac{z}{v})}{Z_o} - \frac{\Gamma_L f(t + \frac{z}{v} - \frac{2l}{v})}{Z_o}$$

we obtain the "extended" voltage and current impulse response functions

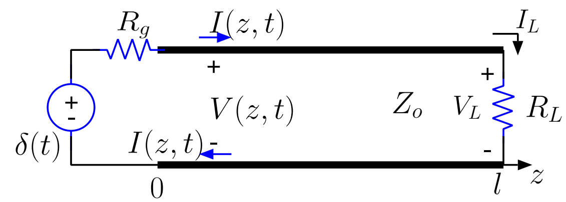
$$V(z,t) = \tau_g \delta(t - \frac{z}{v}) + \Gamma_L \tau_g \delta(t + \frac{z}{v} - \frac{2l}{v}) + \Gamma_g \Gamma_L \tau_g \delta(t - \frac{z}{v} - \frac{2l}{v}) + \Gamma_g \Gamma_L^2 \tau_g \delta(t + \frac{z}{v} - \frac{4l}{v})$$

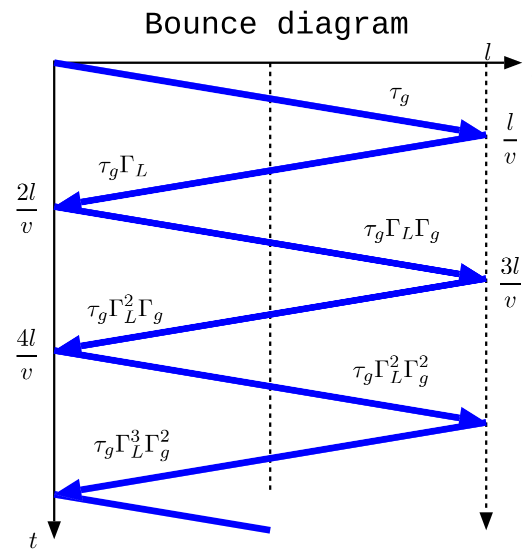
and

$$I(z,t) = Z_o^{-1}[\tau_g \delta(t - \frac{z}{v}) - \Gamma_L \tau_g \delta(t + \frac{z}{v} - \frac{2l}{v}) + \Gamma_g \Gamma_L \tau_g \delta(t - \frac{z}{v} - \frac{2l}{v}) - \Gamma_g \Gamma_L^2 \tau_g \delta(t + \frac{z}{v} - \frac{4l}{v})]$$

respectively.

- At this point the algebra is pretty messy, but a straightforward pattern is emerging (to obviate the need for algebraic analysis for the upcoming epochs) that is best appreciated with the help of **bounce diagrams** explained next:





– A *bounce diagram* is a plot of the "trajectories" of traveling impulses found on transmission line segments excited by impulse inputs.

– The horizontal axis represents position z of the traveling impulses while time t is represented by a downward pointing axis.

– The first slanted line on the top of the diagram, representing the traveling impulse

$$\tau_g \delta(t - \frac{z}{v}),$$

(first term of $h_z(t) = V(z,t)$) is "reflected" at time $t = \frac{\ell}{v}$ from load R_L to turn into a backward propagating impulse

$$\tau_g \Gamma_L \delta(t + \frac{z}{v} - \frac{2\ell}{v})$$

represented by the second line of the diagram.

– The backward propagating impulse reaches $z = 0$ at $t = \frac{2\ell}{v}$ and is reflected once more with a reflection coefficient

$$\Gamma_g = \frac{R_g - Z_o}{R_g + Z_o}$$

to become a forward propagating impulse

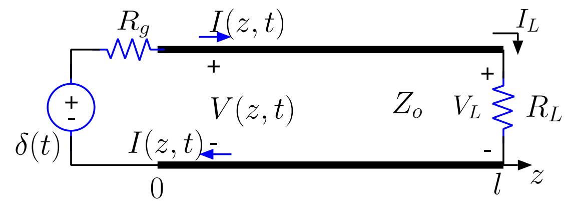
$$\tau_g \Gamma_L \Gamma_g \delta(t - \frac{z}{v} - \frac{2\ell}{v})$$

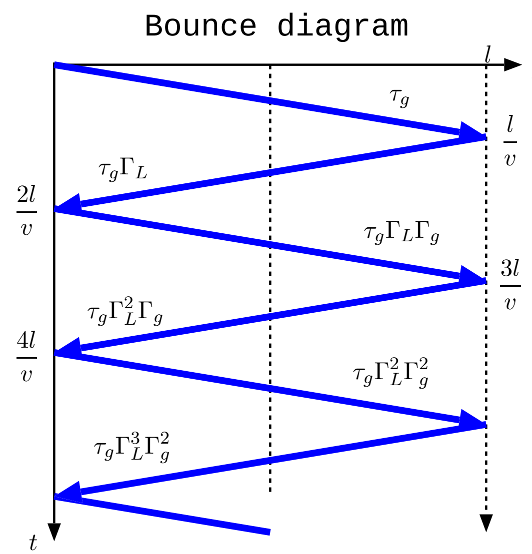
represented by the third line of the diagram.

- Reflection at R_g is in effect the same physical process as reflection at R_L and therefore its coefficient Γ_g is identical with Γ_L except for the replacement of R_L by R_g.

– The bounce diagram is advanced in time with further reflections occurring at both ends.

– We show the calculated weights of traveling impulses directly on the diagram just above the slanted lines representing the trajectories of each traveling impulse (each having a lifetime of ℓ/v)





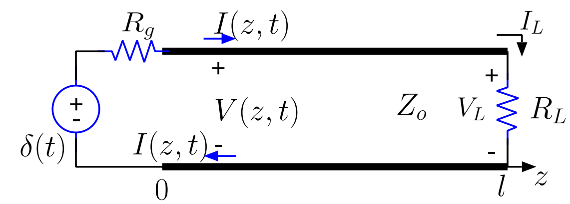
- Using the bounce diagram, the full expressions for the voltage and current impulse response functions of the circuit can be written as

$$\begin{aligned} V(z,t) &= \tau_g \sum_{n=0}^{\infty} (\Gamma_L \Gamma_g)^n \delta(t - \frac{z}{v} - n\frac{2\ell}{v}) \\ &\quad + \tau_g \Gamma_L \sum_{n=0}^{\infty} (\Gamma_L \Gamma_g)^n \delta(t + \frac{z}{v} - (n+1)\frac{2\ell}{v}) \end{aligned}$$

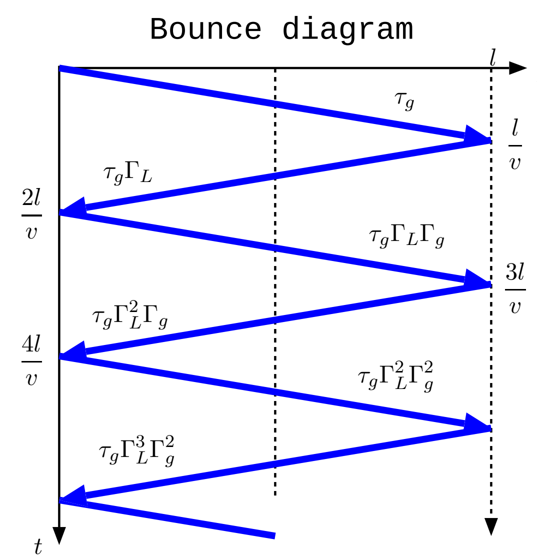
and

$$\begin{aligned} I(z,t) &= \tfrac{\tau_g}{Z_o} \sum_{n=0}^{\infty} (\Gamma_L \Gamma_g)^n \delta(t - \frac{z}{v} - n\frac{2\ell}{v}) \\ &\quad - \frac{\tau_g}{Z_o} \Gamma_L \sum_{n=0}^{\infty} (\Gamma_L \Gamma_g)^n \delta(t + \frac{z}{v} - (n+1)\frac{2\ell}{v}). \end{aligned}$$

- Although these series formulae look daunting, only the lower order terms usually matter — that is true because $|\Gamma_L| \leq 1$ and $|\Gamma_g| \leq 1$ and thus $(\Gamma_L \Gamma_g)^n$ is typically a rapidly diminishing function of n (unless the ckt is "dissipation free" and resonant, a concept explored in Lecture 31).



- We typically rely on the bounce diagram technique more so than the series expressions developed above. This will be illustrated by several examples in the next lecture.



  - The main idea is to combine *delayed* versions of the circuit input $f_i(t)$ with the impulse weights indicated on the bounce diagram, since, in general, the convolution $\delta(t - T_z) * f_i(t) = f_i(t - T_z)$ for any z-dependent delay such as $\frac{z}{v}$, $\frac{z}{v} - \frac{2\ell}{v}$, etc...