25 Wave reflection and transmission

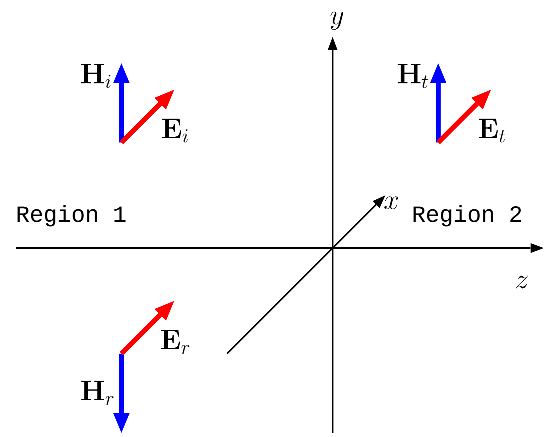
In this lecture we will examine the phenomenon of plane-wave reflections at an interface separating two homogeneous regions where Maxwell's equations allow for traveling TEM wave solutions. The solutions will also need to satisfy the boundary condition equations repeated in the margin. We will consider a propagation scenario in which (see margin):

$$\begin{aligned}\hat{n}\cdot(\mathbf{D}^+ - \mathbf{D}^-) &= \rho_s\\ \hat{n}\cdot(\mathbf{B}^+ - \mathbf{B}^-) &= 0\\ \hat{n}\times(\mathbf{E}^+ - \mathbf{E}^-) &= 0\\ \hat{n}\times(\mathbf{H}^+ - \mathbf{H}^-) &= \mathbf{J}_s\end{aligned}$$

1. Region 1 where $z < 0$ is occupied by a perfect dielectric with medium parameters μ_1, ϵ_1, and $\sigma_1 = 0$,

2. Region 2 where $z > 0$ is homogeneous with medium parameters μ_2, ϵ_2, and σ_2,

3. Interface $z = 0$ contains no surface charge or current except possibly in $\sigma_2 \to \infty$ limit which will be considered separately at the end.



- In Region 1 we envision an **incident plane-wave** with linear-polarized field phasors

$$\tilde{\mathbf{E}}_i = \hat{x}E_o e^{-j\beta_1 z} \quad \text{and} \quad \tilde{\mathbf{H}}_i = \hat{y}\frac{E_o}{\eta_1}e^{-j\beta_1 z},$$

  where

  - E_o is the wave amplitude due to far away source located in $z \to -\infty$ region,
  - $\eta_1 = \sqrt{\frac{\mu_1}{\epsilon_1}}$ and $\beta_1 = \omega\sqrt{\mu_1\epsilon_1}$.

Fields above satisfy Maxwell's equations in Region 1, but if there were no other fields in Regions 1 and 2 **boundary condition equations** requiring continuous tangential **E** and **H** at the $z = 0$ interface would be violated.

In order to comply with the boundary condition equations we postulate a set of reflected and transmitted wave fields in Regions 1 and 2 as follows:

- In Region 1 we postulate a **reflected plane-wave** with linear-polarized field phasors

$$\tilde{\mathbf{E}}_r = \hat{x}\Gamma E_o e^{j\beta_1 z} \quad \text{and} \quad \tilde{\mathbf{H}}_r = -\hat{y}\frac{\Gamma E_o}{\eta_1}e^{j\beta_1 z}$$

  including an unknown Γ that we will refer to as **reflection coefficient**.

  - Note that the reflected wave propagates in $-z$ direction (direction of $\tilde{\mathbf{H}}_r$ and the exponential terms have been adjusted accordingly).

- In Region 2 we postulate a **transmitted plane-wave** with linear-polarized field phasors

$$\tilde{\mathbf{E}}_t = \hat{x}\tau E_o e^{-\gamma_2 z} \quad \text{and} \quad \tilde{\mathbf{H}}_t = \hat{y}\frac{\tau E_o}{\eta_2}e^{-\gamma_2 z}$$

  including an unknown τ that we will refer to as **transmission coefficient**.

  - Note that the transmitted wave propagates in z direction, and
  - since Region 2 is conducting we have

$$\eta_2 = \sqrt{\frac{j\omega\mu_2}{\sigma_2 + j\omega\epsilon_2}}$$

    and

$$\gamma_2 = \sqrt{(j\omega\mu_2)(\sigma_2 + j\omega\epsilon_2)} = \alpha_2 + j\beta_2.$$

Incident:

$$\tilde{\mathbf{E}}_i = \hat{x}E_o e^{-j\beta_1 z},$$

$$\tilde{\mathbf{H}}_i = \hat{y}\frac{E_o}{\eta_1}e^{-j\beta_1 z},$$

Reflected:

$$\tilde{\mathbf{E}}_r = \hat{x}\Gamma E_o e^{j\beta_1 z},$$

$$\tilde{\mathbf{H}}_r = -\hat{y}\frac{\Gamma E_o}{\eta_1}e^{j\beta_1 z},$$

Transmitted:

$$\tilde{\mathbf{E}}_t = \hat{x}\tau E_o e^{-\gamma_2 z},$$

$$\tilde{\mathbf{H}}_t = \hat{y}\frac{\tau E_o}{\eta_2}e^{-\gamma_2 z}.$$

- To determine the unknowns Γ and τ we enforce the following boundary conditions at $z = 0$ where the fields simplify as shown in the margin:

1. **Tangential $\tilde{\mathbf{E}}$ continuous at $z = 0$:** This requires $\tilde{E}_{ix} + \tilde{E}_{rx} = \tilde{E}_{tx}$, leading to

$$(1 + \Gamma)E_o = \tau E_o \quad \Rightarrow \quad \boxed{1+\Gamma=\tau}$$

2. **Tangential $\tilde{\mathbf{H}}$ continuous at $z = 0$:** This requires $\tilde{H}_{iy} + \tilde{H}_{ry} = \tilde{H}_{ty}$, leading to

$$(1 - \Gamma)\frac{E_o}{\eta_1} = \tau\frac{E_o}{\eta_2} \quad \Rightarrow \quad \boxed{1\text{-}\Gamma=\frac{\eta_1}{\eta_2}\tau}$$

Replacing τ by $1 + \Gamma$ in the second equation, we can solve for the reflection coefficient as

$$\boxed{\Gamma=\frac{\eta_2-\eta_1}{\eta_2+\eta_1}}$$

and substituting this in turn in the first equation we can solve for the transmission coefficient as

$$\boxed{\tau=\frac{2\eta_2}{\eta_2+\eta_1}}$$

The results are summarized in the margin on the next page.

Incident at $z = 0$:

$$\tilde{\mathbf{E}}_i = \hat{x}E_o, \quad \tilde{\mathbf{H}}_i = \hat{y}\frac{E_o}{\eta_1},$$

Reflected at $z = 0$:

$$\tilde{\mathbf{E}}_r = \hat{x}\Gamma E_o, \quad \tilde{\mathbf{H}}_r = -\hat{y}\frac{\Gamma E_o}{\eta_1}$$

Transmitted at $z = 0$:

$$\tilde{\mathbf{E}}_t = \hat{x}\tau E_o, \quad \tilde{\mathbf{H}}_t = \hat{y}\frac{\tau E_o}{\eta_2}$$

Special cases:

1. **Region 2 is a perfect conductor with $\sigma_2 \to \infty$:** In that case $\eta_2 \to 0$, and consequently

$$\Gamma = -1 \quad \text{and} \quad \tau = 0.$$

Incident wave cannot penetrate the **perfect conductor**, and it reflects totally back into Region 1 — we will study this idealized limiting case more carefully later on.

Practical application of total reflection: mirrors

2. **Region 2 is the same as Region 1:** In that case $\eta_2 = \eta_1$, and consequently

$$\Gamma = 0 \quad \text{and} \quad \tau = 1.$$

This is the **matched impedance** case when no reflection takes place and the incident wave is transmitted in its entirety.

3. **Region 2 is lossless, i.e., $\sigma_2 = 0$:** Unless $\eta_2 = \eta_1$ there will be reflected as well as transmitted waves.

Partial reflections can be reduced by applying a "anti-glare" coating[1] on the surface, a practice known as "impedance matching".

[1]This is a $\lambda/4$ thick layer of a material having a characteristic impedance given by $\sqrt{\eta_1\eta_2}$ — the reason for why this "quarter-wave matching" works will be discussed when we study transmission lines later on.

Reflection coeff.:

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1},$$

Transmission coeff.:

$$\tau = \frac{2\eta_2}{\eta_2 + \eta_1}.$$

Memorize the Γ formula, and memorize τ as "one plus Γ".

Above,

$$\eta_1 = \sqrt{\frac{\mu_1}{\epsilon_1}}$$

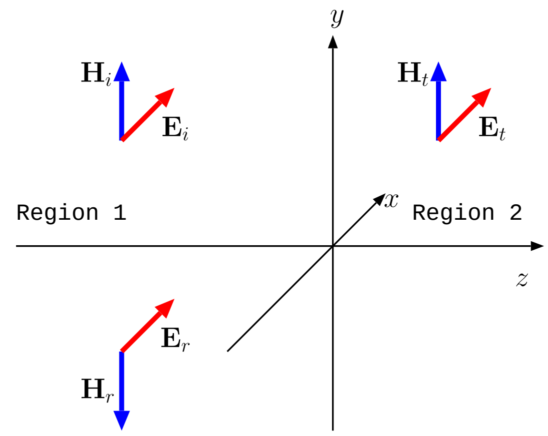
and

$$\eta_2 = \sqrt{\frac{j\omega\mu_2}{\sigma_2 + j\omega\epsilon_2}}.$$

Example 1: An plane-wave in vacuum,

$$\tilde{\mathbf{E}}_i = \hat{x}\sqrt{120\pi}e^{-j\beta_1 z} \ \frac{V}{m},$$

is incident at $z = 0$ on a dielectric medium with $\mu = \mu_o$ and $\epsilon = \frac{9}{4}\epsilon_o$. Determine the average Poynting vectors $\langle \mathbf{S}_i \rangle$, $\langle \mathbf{S}_r \rangle$, and $\langle \mathbf{S}_t \rangle$ of the incident, reflected, and transmitted fields.



Solution: The intrinsic impedance of the second medium occupying $z > 0$ is

$$\eta_2 = \sqrt{\frac{\mu_o}{\frac{9}{4}\epsilon_o}} = \frac{2}{3}\eta_o.$$

Therefore, the reflection coefficient is

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{\frac{2}{3}\eta_o - \eta_o}{\frac{2}{3}\eta_o + \eta_o} = \frac{\frac{2}{3} - 1}{\frac{2}{3} + 1} = \frac{2-3}{2+3} = -\frac{1}{5}$$

and the transmission coefficient is

$$\tau = 1 + \Gamma = 1 - \frac{1}{5} = \frac{4}{5}.$$

The reflected wave therefore has the field phasors

$$\tilde{\mathbf{E}}_r = -\frac{1}{5}\hat{x}\sqrt{120\pi}e^{j\beta_1 z} \ \text{ and } \ \tilde{\mathbf{H}}_r = \frac{1}{5\eta_o}\hat{y}\sqrt{120\pi}e^{j\beta_1 z}$$

and

$$\langle \mathbf{S}_r \rangle = \frac{1}{2}\text{Re}\{\tilde{\mathbf{E}}_r \times \tilde{\mathbf{H}}_r^*\} = -\hat{z}\frac{1}{2}(\frac{1}{5})^2\frac{120\pi}{\eta_o} \approx -\hat{z}\frac{1}{2}(\frac{1}{5})^2\,\frac{\text{W}}{\text{m}^2}.$$

The transmitted wave, likewise, has the field phasors

$$\tilde{\mathbf{E}}_t = \frac{4}{5}\hat{x}\sqrt{120\pi}e^{-j\beta_2 z} \quad \text{and} \quad \tilde{\mathbf{H}}_t = \frac{4}{5\frac{2}{3}\eta_o}\hat{y}\sqrt{120\pi}e^{-j\beta_2 z}$$

and

$$\langle \mathbf{S}_t \rangle = \frac{1}{2}\text{Re}\{\tilde{\mathbf{E}}_t \times \tilde{\mathbf{H}}_t^*\} = \hat{z}\frac{1}{2}(\frac{4}{5})^2\frac{3}{2}\frac{120\pi}{\eta_o} \approx \hat{z}\frac{1}{2}(\frac{4}{5})^2\frac{3}{2}\frac{\text{W}}{\text{m}^2}.$$

As for the incident wave

$$\tilde{\mathbf{E}}_i = \hat{x}\sqrt{120\pi}e^{-j\beta_1 z} \quad \text{and} \quad \tilde{\mathbf{H}}_i = \frac{1}{\eta_o}\hat{y}\sqrt{120\pi}e^{-j\beta_1 z}$$

and

$$\langle \mathbf{S}_i \rangle = \frac{1}{2}\text{Re}\{\tilde{\mathbf{E}}_i \times \tilde{\mathbf{H}}_i^*\} = \hat{z}\frac{1}{2}\frac{120\pi}{\eta_o} \approx \hat{z}\frac{1}{2}\frac{\text{W}}{\text{m}^2}.$$

Note: We have

$$|\langle \mathbf{S}_r \rangle| + |\langle \mathbf{S}_t \rangle| = \frac{1}{2}(\frac{1}{25} + \frac{16}{25}\frac{3}{2}) = \frac{1}{2}(\frac{1}{25} + \frac{24}{25}) = \frac{1}{2} = |\langle \mathbf{S}_i \rangle|$$

in compliance with energy conservation (as expected) — **energy flux per unit area of the transmitted and reflected waves add up the that of the incident wave!**