18 Wave equation and plane TEM waves in source-free media

With this lecture we start our study of the full set of Maxwell's equations shown in the margin by first restricting our attention to *homogeneous* and *non-conducting* media with constant ϵ and μ and zero σ.

$$\begin{aligned}\nabla \cdot \mathbf{D} &= \rho \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \times \mathbf{H} &= \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}.\end{aligned}$$

- Our first objective is to show that non-trivial (i.e., non-zero) time-varying field solutions of these equations can be obtained even in the absence of ρ and $\mathbf{J}$.

  – We already know *static* ρ and $\mathbf{J}$ to be the **source** of *static* electric and magnetic fields.

  – We will come to understand that time varying ρ and $\mathbf{J}$, which necessarily obey the continuity equation

  $$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0,$$

  constitute the **source** of *time-varying electromagnetic* fields.

  Despite these intimate connections between the sources ρ and $\mathbf{J}$ and the fields

  $$\mathbf{D} = \epsilon \mathbf{E} \quad \text{and} \quad \mathbf{B} = \mu \mathbf{H},$$

  non-trivial field solutions can exist in source-free media as we will see shortly.

- Such field solutions in fact represent electromagnetic waves, a familiar example of which is **light**.
- Another example is **radiowaves** that we use when we communicate using wireless devices such as radios, cell-phones, WiFi, etc.
- Different types of electromagnetic waves are distinguished by their oscillation frequencies, and include

  - radiowaves,
  - microwaves,
  - infrared,
  - light,
  - ultraviolet,
  - X-rays, and gamma rays,

  going across the **electromagnetic spectrum** from low to high frequencies.

  We are well aware that these types of electromagnetic waves can travel across empty regions of space — e.g., from sun to Earth — transporting energy and heat as well as momentum.

  - Next, we will discover their general properties by examining Maxwell's equations under the restriction $\rho = \mathbf{J} = 0$.

- In source-free and homogeneous regions where $\rho = \mathbf{J} = 0$ and ϵ and μ are constant, we can simplify Maxwell's equations as shown in the margin.

  - If there are non-trivial solutions of these equations, namely $\mathbf{E}(\mathbf{r}, t) \neq 0$ and $\mathbf{H}(\mathbf{r}, t) \neq 0$, they evidently need to be divergence-free.
  - They also have to be "curly" according to the last two equations: Faraday's and Ampere's laws.

$$\begin{aligned} \nabla \cdot \mathbf{E} &= 0 \\ \nabla \cdot \mathbf{H} &= 0 \\ \nabla \times \mathbf{E} &= -\mu \frac{\partial \mathbf{H}}{\partial t} \\ \nabla \times \mathbf{H} &= \epsilon \frac{\partial \mathbf{E}}{\partial t}. \end{aligned}$$

- Next we will make use of vector identity

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}$$

which should be familiar from an earlier homework problem.

  - Since the electric field $\mathbf{E}$ is divergence-free in the absence of sources, this identity simplifies as

$$\nabla \times (\nabla \times \mathbf{E}) = -\nabla^2 \mathbf{E}$$

  where in the right side $\nabla^2 \mathbf{E}$ is the Laplacian of $\mathbf{E}$.
  - Using this result we can express the curl of Faraday's law as

$$\nabla \times [\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t}] \;\; \Rightarrow \;\; -\nabla^2 \mathbf{E} = -\mu \frac{\partial}{\partial t} \nabla \times \mathbf{H},$$

  which combines with the Ampere's law to produce

$$\nabla^2 \mathbf{E} = \mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2},$$

which can be written explicitly as

3D vector wave equation

$$\frac{\partial^2 \mathbf{E}}{\partial x^2} + \frac{\partial^2 \mathbf{E}}{\partial y^2} + \frac{\partial^2 \mathbf{E}}{\partial z^2} = \mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}.$$

Recall that our objective is to see whether a non-trivial time-varying solution of Maxwell's equations can exist in source-free media.

Our objective at this stage is not finding a general solution; it is instead identifying a simple example of a non-trivial time-varying $\mathbf{E}(\mathbf{r}, t)$, if we can.

For example, can a field solution

$$\mathbf{E}(\mathbf{r}, t) = \hat{x} E_x(z, t)$$

that only depends on z and t and "polarized" in x-direction exist? If it can exist, what would be the properties of this x-**polarized** solution?

- To find out, we note that with $\mathbf{E} = \hat{x} E_x(z, t)$, the above "wave equation" is reduced to

1D scalar wave equation

$$\frac{\partial^2 E_x}{\partial z^2} = \mu\epsilon \frac{\partial^2 E_x}{\partial t^2},$$

an equation that is known as a **1D scalar wave equation**, as opposed to the **3D *vector* wave equation** above.

  - Now, by substitution, we can easily show that

$$E_x = \cos(\omega(t - \sqrt{\mu\epsilon} z)),$$

satisfies the 1D wave equation and represents an x-polarized time-periodic field solution with an **oscillation frequency** ω.

– 1D wave equation can also be satisfied by

$$E_x = \cos(\omega(t + \sqrt{\mu\epsilon}z)).$$

Let us jointly refer to these solutions as

$$E_x = \cos(\omega(t \mp \frac{z}{v})),$$

where

$$v \equiv \frac{1}{\sqrt{\mu\epsilon}}$$

has the dimensions of m/s (i.e., velocity) and the algebraic signs $\mp$ distinguish between the "travel directions" of these *possible* "wave solutions" as elaborated later on.

- Let us next find out the magnetic field intensity **H** that accompanies the x-polarized electric field wave solution

$$\mathbf{E} = \hat{x}\cos(\omega(t \mp \frac{z}{v})).$$

– Since the curl of **E** is

$$\nabla \times \mathbf{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & 0 & 0 \end{vmatrix} = \hat{y}\frac{\partial E_x}{\partial z} = \pm\hat{y}\sin(\omega(t \mp \frac{z}{v}))\frac{\omega}{v},$$

Faraday's law

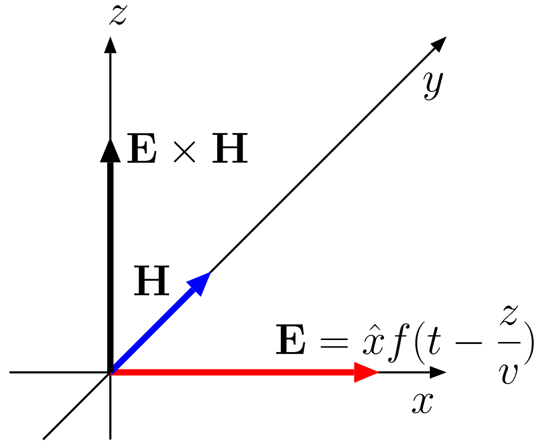
$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t}$$

requires that $\mathbf{H}$ should satisfy

$$-\mu \frac{\partial \mathbf{H}}{\partial t} = \pm \hat{y} \sin(\omega(t \mp \frac{z}{v})) \frac{\omega}{v}.$$

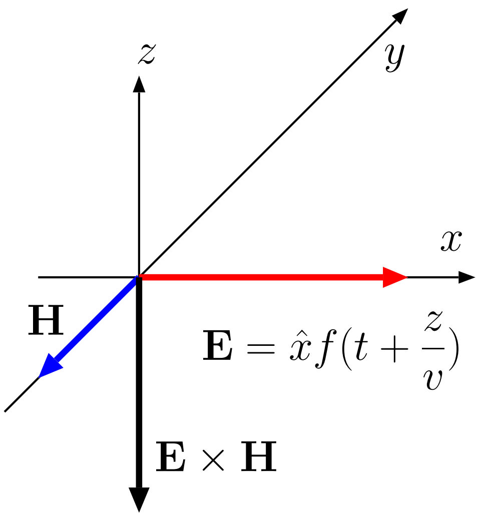
Finding the *time-dependent* anti-derivative (and remembering $v = 1/\sqrt{\mu\epsilon}$), we obtain

$$\mathbf{H} = \pm \hat{y} \sqrt{\frac{\epsilon}{\mu}} \cos(\omega(t \mp \frac{z}{v})).$$



- The results above, namely our x-polarized non-trivial field solutions of Maxwell's equations in source-free homogeneous space, can be represented more compactly as

$$\mathbf{E} = \hat{x} f(t \mp \frac{z}{v}) \quad \text{and} \quad \mathbf{H} = \pm \hat{y} \frac{f(t \mp \frac{z}{v})}{\eta},$$



where

$$f(t) \equiv \cos(\omega t) = \mathrm{Re}\{e^{j\omega t}\} = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$$

is the field **waveform**,

$$\eta \equiv \sqrt{\frac{\mu}{\epsilon}}$$

is known as **intrinsic impedance** (and measured in units of ohms).

- Since Maxwell's equations with constant μ and ϵ are linear and time-invariant (LTI), the field solutions above can be further generalized by using their weighted and time-shifted superpositions such as

$$f(t) = \sum_n A_n \cos(\omega_n t + \theta_n)$$

and

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega$$

having frequency dependent weighting factors A_n and $F(\omega)$. And since according to Fourier analysis all practical signals $f(t)$ can be synthesized in these forms, it follows that the field solutions above are valid with *arbitrary* waveforms $f(t)$.

d'Alembert wave solutions

Solutions

$$\mathbf{E}, \mathbf{H} \propto f(t \mp \frac{z}{v})$$

of the 1D scalar wave equation with arbitrary $f(t)$ are known as **d'Alembert wave solutions**.

- d'Alembert solution

$$\mathbf{E}, \mathbf{H} \propto f(t - \frac{z}{v})$$

describes electromagnetic waves traveling in $+z$ direction, whereas solution

$$\mathbf{E}, \mathbf{H} \propto f(t + \frac{z}{v})$$

describes electromagnetic waves traveling in $-z$ direction (see margin).

In each case the travel speed is

$$v = \frac{1}{\sqrt{\mu\epsilon}} \xrightarrow{\text{free space}} \frac{1}{\sqrt{\mu_o\epsilon_o}} \equiv c \approx 3 \times 10^8\,\text{m/s}.$$

- $\mathbf{H}$ solution can be obtained from $\mathbf{E}$ by dividing it with η and rotating it by $90°$ so that vector $\mathbf{E} \times \mathbf{H}$ points in direction the waves travel.

- $\mathbf{E}$ can be obtained from $\mathbf{H}$ by multiplying it with η and rotating it by $90°$ so that vector $\mathbf{E} \times \mathbf{H}$ — called **Poynting vector** — once again points in direction the waves travel.

In each case the intrinsic impedance is

$$\eta = \sqrt{\frac{\mu}{\epsilon}} \xrightarrow{\text{free space}} \sqrt{\frac{\mu_o}{\epsilon_o}} \equiv \eta_o \approx 120\pi\,\text{ohms}.$$

Transformation rules above also hold for y-**polarized** wave solutions

$$\mathbf{E} = \hat{y} f(t \mp \frac{z}{v}) \quad \text{and} \quad \mathbf{H} = \mp\hat{x}\frac{f(t \mp \frac{z}{v})}{\eta}.$$

Question: What about z-**polarized** waves

$$\mathbf{E} = \hat{z} f(t \mp \frac{z}{v}),$$

can they exist?

Answer: No, z-polarized waves $\hat{z} f(t \mp \frac{z}{v})$ traveling in $\pm z$ direction cannot exist because they would violate the divergence-free condition $\nabla \cdot \mathbf{E} = 0$.