17 Magnetization current, Maxwell's equations in material media

- Consider the microscopic-form Maxwell's equations

$$
\begin{aligned}
\nabla \cdot \mathbf{D} &= \rho \qquad \text{Gauss's law} \\
\nabla \cdot \mathbf{B} &= 0 \\
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \qquad \text{Faraday's law} \\
\nabla \times \mathbf{H} &= \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}, \qquad \text{Ampere's law}
\end{aligned}
$$

  where

$$
\begin{aligned}
\mathbf{D} &= \epsilon_o \mathbf{E} \\
\mathbf{B} &= \mu_o \mathbf{H}.
\end{aligned}
$$

- Direct applications of these equations in material media containing a colossal number of bound charges is impractical.

- Macroscopic-form Maxwell's equations suitable for material media are obtained by first expressing ρ and $\mathbf{J}$ above as the macroscopic quantities

$$\rho = \rho_f - \nabla \cdot \mathbf{P}$$

  and

$$\mathbf{J} = \mathbf{J}_f + \frac{\partial \mathbf{P}}{\partial t} + \nabla \times \mathbf{M}$$

  where

– subscripts f indicate charge and current density contributions due to free charge carriers,

– the term $-\nabla \cdot \mathbf{P}$ denotes the **bound charge density**,

– the term $\frac{\partial \mathbf{P}}{\partial t}$ denotes the **polarization current density** due to oscillating dipoles (already discussed in Lecture 11), and

– $\nabla \times \mathbf{M}$ is a **"magnetization current density"** also due to bound charges, an effect that we will discuss and clarify later in this section.

Using these expressions in Gauss's and Ampere's laws

$$\begin{aligned} \nabla \cdot \epsilon_o \mathbf{E} &= \rho \qquad \text{Gauss's law} \\ \nabla \times \mu_o^{-1} \mathbf{B} &= \mathbf{J} + \frac{\partial \epsilon_o \mathbf{E}}{\partial t}, \qquad \text{Ampere's law} \end{aligned}$$

we obtain

$$\begin{aligned} \nabla \cdot (\epsilon_o \mathbf{E} + \mathbf{P}) &= \rho_f \qquad \text{Gauss's law} \\ \nabla \times (\mu_o^{-1} \mathbf{B} - \mathbf{M}) &= \mathbf{J}_f + \frac{\partial}{\partial t}(\epsilon_o \mathbf{E} + \mathbf{P}), \qquad \text{Ampere's law.} \end{aligned}$$

Now, re-define $\mathbf{D}$ and $\mathbf{H}$ as

$$\mathbf{D} = \epsilon_e \mathbf{E} + \mathbf{P} = \epsilon \mathbf{E}$$

and

$$\mathbf{H} = \mu_o^{-1} \mathbf{B} - \mathbf{M} = \mu^{-1} \mathbf{B},$$

respectively, and drop the subscripts f which will no longer be needed.

Using these new definitions, the full set of Maxwell's equations now read as (the same form as before)

$$\begin{aligned}
\nabla \cdot \mathbf{D} &= \rho \qquad \text{Gauss's law} \\
\nabla \cdot \mathbf{B} &= 0 \\
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \qquad \text{Faraday's law} \\
\nabla \times \mathbf{H} &= \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}, \qquad \text{Ampere's law}
\end{aligned}$$

with

$$\begin{aligned}
\mathbf{D} &= \epsilon \mathbf{E} \\
\mathbf{B} &= \mu \mathbf{H},
\end{aligned}$$

where ρ and $\mathbf{J}$ are understood to be due to free charge carriers only.

- We had already seen many aspects of the above procedure for obtaining the macroscopic form field equations earlier on (e.g., in Lectures 8 and 11).
  - In particular we were already familiar with the revised definition of $\mathbf{D} = \epsilon \mathbf{E}$ along with the concept of **medium permittivity** ϵ.
  - The new feature above that requires further discussions is the relation $\mathbf{B} = \mu \mathbf{H}$ along with the concept of **medium permeability** μ. The details of this relation are connected to the concept of "magnetization current" which we discuss next.

- Just like “free charge” density and currents, “bound charge” densities and currents also have to satisfy the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0.$$

  – This equation is automatically satisfied if we substitute

$$\rho = \rho_b = -\nabla \cdot \mathbf{P}$$

  and

$$\mathbf{J} = \mathbf{J}_b = \frac{\partial \mathbf{P}}{\partial t}$$

  in it.

  Verification:

$$\frac{\partial \rho_b}{\partial t} + \nabla \cdot \mathbf{J}_b = \frac{\partial}{\partial t}(-\nabla \cdot \mathbf{P}) + \nabla \cdot \frac{\partial \mathbf{P}}{\partial t} = 0$$

  since the order of time derivative and divergence can be exchanged on the right.

  – But the same equation is also satisfied if we take

$$\mathbf{J}_b = \frac{\partial \mathbf{P}}{\partial t} + \nabla \times \mathbf{M}$$

  for any vector field $\mathbf{M}$ simply because vector $\nabla \times \mathbf{M}$ is divergence free.

Consequently, it is not sufficient to represent bound currents in material media as simply $\frac{\partial \mathbf{P}}{\partial t}$, if bound carriers can also conduct divergence-free currents due to closed-loop orbits.

– In fact, electrons "orbiting" atomic nuclei certainly produce such divergence-free current loops at microscopic scales — we account for such currents at macroscopic scales by including a magnetization current term $\nabla \times \mathbf{M}$ in $\mathbf{J}_b$.

– Also, bound charge motions *within* nucleons[1] — proton and neutrons — produce magnetization currents $\nabla \times \mathbf{M}$.

– Even bare electrons can produce magnetization currents $\nabla \times \mathbf{M}$ because of their *intrinsic spin*[2].

Once $\nabla \times \mathbf{M}$ is included in $\mathbf{J}_b$, it follows from Ampere's law that

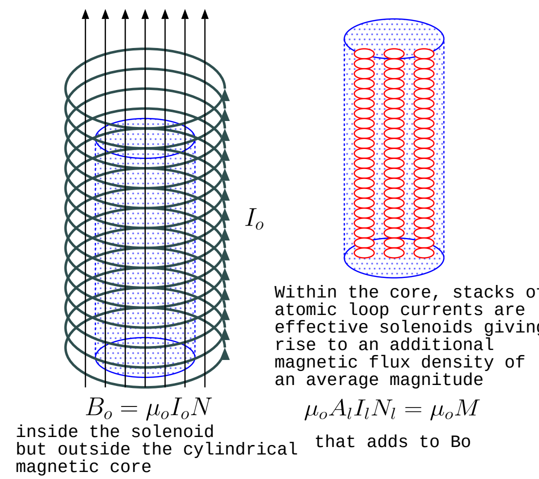
$$\mathbf{H} = \mu_o^{-1}\mathbf{B} - \mathbf{M}$$

where $\mathbf{M}$ is referred to as **magnetization** field.

[1]Physical models of nucleons involve bound charge carriers known as *quarks* which cannot be observed in a free state.

[2]All elementary charge carriers carry an intrinsic magnetization proportional to charge-to-mass ratio $\frac{q}{m}$ and a "spin angular momentum" having quantized values of $\pm\frac{\hbar}{2}$ N.m.s in any measurement direction. Using Heisenberg's uncertainty principle, $\Delta p \Delta r \geq \frac{\hbar}{2}$, we can interpret the *spin angular momentum* of an elementary particle as the lower bound of $\Delta p \Delta r$, the product of quantum uncertainties in particle momentum and position. There is no classical interpretation of spin angular momentum for point particles.

- To get a physical picture about magnetization $\mathbf{M}$ and the physical origin of $\mathbf{H} = \mu_o^{-1}\mathbf{B} - \mathbf{M}$ consider a solenoid wound around some cylindrical shaped material as shown in the margin. We know that with a solenoid current I_o, we would have $\mathbf{H}_o = NI_o\hat{z}$ in the interior of a solenoid with N loops per unit length aligned with the z-axis, and a corresponding magnetic flux density $\mathbf{B}_o = \mu_o NI_o\hat{z}$ when the solenoid core is free space. This will be modified to some $\mathbf{B} = \mathbf{B}_o + \mu_o\mathbf{M}$ when a material core is introduced into the same space, where $\mu_o\mathbf{M}$ stands for the (additional) macroscopic (space averaged) magnetic flux density produced by microscopic current loops localized within the atoms constituting the core.



  - If there are $N_a = \frac{1}{\Delta x \Delta y \Delta z}$ atoms per unit volume in the core, with Δx separations in x direction and so forth, loop currents I_l of a stack of atoms with Δz separations in z would produce an effective solenoid an internal z-directed magnetic flux density of $\mu_o \frac{I_l}{\Delta z}\hat{z}$ and zero exterior field.
  - Since one such atomic stack solenoid with a loop area of A_l will be found for every $\Delta x \Delta y$ cross-sectional area of the core, a macroscopic average magnetic flux density produced by these atomic solenoids would be calculated as (this calculation is similar to finding the average polarization field in a dielectric as discussed in Lecture 8) $\frac{A_l}{\Delta x \Delta y} \times \mu_o \frac{I_l}{\Delta z}\hat{z} = \mu_o N_a I_l A_l \hat{z} \equiv \mu_o \mathbf{M}$, with $\mathbf{M} = N_a \mathbf{m}$, $\mathbf{m} \equiv I_l A_l \hat{z}$.

  - Here $\mathbf{m}$ is the magnetic dipole moment of each current loop (analogous to electric dipole $\mathbf{p} = q\mathbf{r}$), $\mathbf{M}$ is the magnetization field vector (analogous to $\mathbf{P} = N_a\mathbf{p}$), which is a simple product of $\mathbf{m}$ per magnetized atom and the atomic number density N_a in the core.

- Superposing the magnetic flux densities of $\mu_o\mathbf{M}$ and$\mathbf{B}_o$, we obtain $\mathbf{B} = \mathbf{B}_o+\mu_o\mathbf{M}$ for the core region, or for any region of space having a non-zero magnetization $\mathbf{M}$, which then leads to the general result $\mathbf{H} = \mu_o^{-1}\mathbf{B} - \mathbf{M}$, which is further discussed below.
- Notice, whether the flux density $\mathbf{B} = \mathbf{B}_o+\mu_o\mathbf{M}$ inside the material medium is stronger or weaker in magnitude than $\mathbf{B}_o$ depends on the direction of $\mathbf{M}$, which, in turn, depends on the algebraic sign of microscopic loop currents I_l introduced above.
  - Negative I_l is found in **diamagnetic** materials where $|\mathbf{B}| < |\mathbf{B}_o|$, while positive I_l in **paramagnetic** and **ferromagnetic** materials where $|\mathbf{B}| > |\mathbf{B}_o|$, as discussed below.
- Also, the expression $\mathbf{H} = \mu_o^{-1}\mathbf{B} - \mathbf{M}$ leads to $\mathbf{H} = \mu_o^{-1}\mathbf{B} = \mathbf{H_o}$ in the exterior region where $\mathbf{M} = 0$, indicating that while fields $\mathbf{B}$ and $\mathbf{B_o}$if the interior and exterior are different, $\mathbf{H}$ is the same in both regions (analogous with $\mathbf{D}$ in dielectrics).

- Lab measurements — e.g., inductances L measured for coils wound around magnetic materials[3] — show that for a large class of materials

$$\mathbf{M} \equiv \mu_o^{-1}\mathbf{B} - \mathbf{H}$$

varies linearly with $\mathbf{H}$ (which is of course possible only when $\mathbf{B}$ also varies linearly with $\mathbf{H}$).

  - In that case we write

$$\mathbf{M} = \chi_m \mathbf{H},$$

where χ_m is a dimensionless parameter called **magnetic susceptibility**, and obtain a relation

Magnetic susceptibility and permeability

$$\mathbf{B} = \mu_o(1 + \chi_m)\mathbf{H} = \mu\mathbf{H},$$

where

$$\mu = \mu_o(1 + \chi_m)$$

is called the **permeability** of the medium.

[3] Recall from Lecture 15 that $L \propto \mu$ when inductors are wound around materials with permeability μ.

- For a large class of materials with $\mathbf{M} \propto \mathbf{H}$, it is observed that $|\chi_m| \ll 1$. In that case, the material is called

  - **Diamagnetic** if $\chi_m < 0$:

    - Diamagnetism occurs when an applied magnetic field *induces* electron **orbital angular momentum** in a collection of atoms having no net permanent magnetization $\mathbf{M}$ — in such materials electron clouds around atomic nuclei spin up in accordance with Lenz's to generate magnetic fields opposing the applied magnetic field so as to keep $\mathbf{B} = \mu\mathbf{H}$ smaller than $\mu_o\mathbf{H}$. This happens in materials that we ordinarily think of being non-magnetic (wood, glass, water, etc.). Diamagnetic materials are in fact very weakly repulsed by permanent magnets since $\mu \approx \mu_o$ in all diamagnetic materials.

  - **Paramagnetic** if $\chi_m > 0$:

    - Paramagnetism occurs in materials composed of atoms having permanent magnetic dipole moments due to electron **spin angular momentum** — magnetic dipoles of such atoms co-align with the applied magnetic field due to $\mathbf{v} \times \mathbf{B}$ related torques, leading to $\mathbf{M}$ pointing in the applied $\mathbf{B}$ direction[4]. This happens for atoms with unfilled inner electron shells, because in filled shells electron spins are opposite (due to Pauli

[4]In these materials the described paramagnetism overcomes the diamagnetic tendency of the material caused by the orbital angular momenta of its electrons around atomic nuclei.

exclusion principle) and cancel one another. Unfilled outer shells do not usually give rise to paramagnetism because interactions between adjacent atoms in that case give rise to opposite spins of their outer shell electrons. Paramagnetic materials are very weakly attracted to permanent magnets (e.g., aluminum, lithium, tungsten).

- In a small class of materials known as **ferromagnets** — iron, nickel, and cobalt, which are metals with atoms having unfilled inner electron shells, and their various alloys — $\mathbf{M}$ can arise spontaneously (because permanent magnetic dipole moments of nearby atoms produced by **electron spins** become co-aligned as a consequence of conduction electrons moving through the lattice) and turns out to be a non-linear function of present and past values of $\mathbf{H}$, in which case experimentally obtained relations, denoted as

$$\mathbf{B} = \mathbf{B}(\mathbf{H}),$$

  need to be used in Maxwell's equations. It is even possible to have non-zero $\mathbf{B}$ in such materials with zero $\mathbf{H}$ — permanent magnets have that property.

- First principles modeling of χ_m or the $\mathbf{B} = \mathbf{B}(\mathbf{H})$ relation requires quantum mechanics (classical models turn out to be not accurate enough). Overall, the models give rise to frequency dependent results, involving loss as well as resonance features (also exhibited in Lorentz-Drude models of χ_e examined in Lecture 11) relevant for applications including various magnetic imaging techniques.