15 Inductance — coil, solenoid, shorted coax

- Given a circular coil with some resistance R and conducting some current I, the magnetic flux Ψ produced by I and "linking" the coil itself — see figure on the right — can be expressed as

$$\Psi = LI$$

using a non-negative proportionality constant

$$L = \frac{\Psi}{I}$$

termed the ***self*-inductance** of the coil measured in units of Henries (H=Wb/A)[1].

- Given $\Psi = LI$, and its time derivative

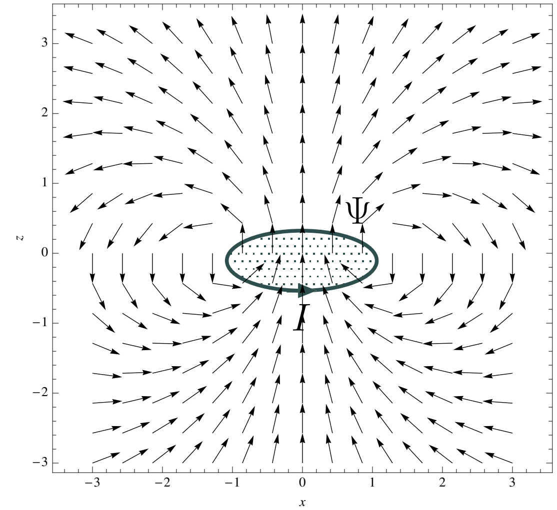
$$\frac{d\Psi}{dt} = L\frac{dI}{dt},$$

it follows that Faraday's equation applied to the coil is

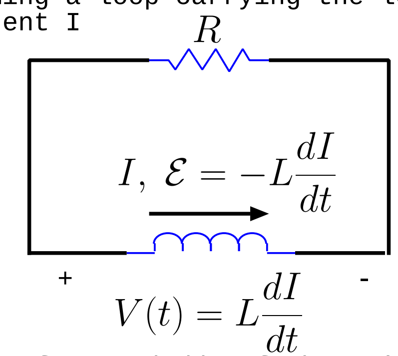
$$\mathcal{E} = -\frac{d\Psi}{dt} = -L\frac{dI}{dt},$$

indicating a ***self*-emf** $-L\frac{dI}{dt}$ representing a ***voltage rise*** around the coil in the direction of current flow $I = \mathcal{E}/R$ — see an equivalent circuit model for the coil derived from these relations shown on the right.

(a) A one turn coil with current I generates its own linked magnetic flux LI as shown, where a non-negative L is the inductance of the coil.



(b) An equivalent circuit model for the coil expressed in terms of lumped resistor R and inductor L forming a loop carrying the loop current I



The emf RI=-LdI/dt of the coil appears as a voltage rise across the inductor in the ckt model, as well as a voltage drop across the resistor, both taken in the direction of current I. Voltage drop V across the inductor in the current direction is LdI/dt, as we learned in our circuit courses.

[1] As opposed to a *mutual* inductance M, also measured in Henries, relating the flux linking a coil C to a current I_o flowing in a second coil C_o.

– The current I and self-emf $\mathcal{E}$ are then the solutions of differential equations

$$RI = -L\frac{dI}{dt} \text{ and } R\mathcal{E} = -L\frac{d\mathcal{E}}{dt},$$

respectively, and exhibit an exponential decay with a time constant of $\tau = L/R$ (just like in LR circuits seen in ckt courses, and in analogy with time constant $\tau = RC$ that governs voltage decays in RC circuits).

  ◦ Note that $\tau = L/R$ implies that when the inductance L is large, so is time constant τ, and current decay in the inductor is slow — inductors with large L will behave like slowly time-varying current sources (just like capacitors behaving like time-varying voltage sources) as they release their stored energy (while maintaining a voltage rise $-L\frac{dI}{dt}$ determined by other elements in their connected circuits).

• For an inductor consisting of n-loops, the emf $\mathcal{E}$ measured across all n-loops is naturally (since n emf's add up)

$$\mathcal{E} = n(-\frac{d}{dt}\Psi) = -\frac{d}{dt}n\Psi \equiv -L\frac{dI}{dt}$$

implying an inductance

$$L \equiv \frac{n\Psi}{I}.$$

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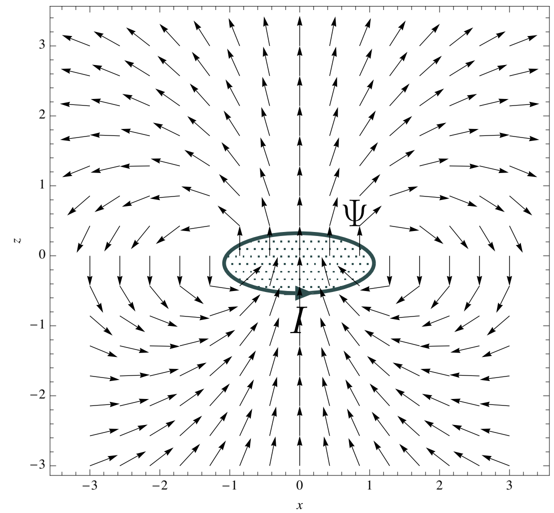
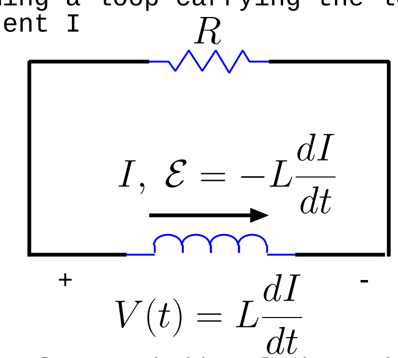


(b) An equivalent circuit model for the coil expressed in terms of lumped resistor R and inductor L forming a loop carrying the loop current I



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Example 1: An n-turn coil has a resistance $R = 1\,\Omega$ and inductance of $1\,\mu$H. If it is conducting 3 A current at $t = 0$, determine $I(t)$ for $t > 0$.

Solution: Current flow in the resistive n-turn coil will be driven by self-emf $\mathcal{E} = -L\frac{dI}{dt}$ matching a voltage drop RI. Hence

$$RI = -L\frac{dI}{dt} \;\leftrightarrow\; \frac{dI}{dt} + \frac{R}{L}I = 0 \;\Rightarrow\; I(t) = I(0)e^{-\frac{R}{L}t} = 3e^{-10^6 t}\text{ A.}$$

- As illustrated by above example, current I around a resistive loop C will in general be obtained by solving a *differential equation* constructed using the emf of the loop.
  - The algebraic $I = \frac{\mathcal{E}}{R}$ solution used last lecture assumed that self-emf $-L\frac{dI}{dt}$ produced by the induced current $I(t)$ is small compared to an externally produced emf.

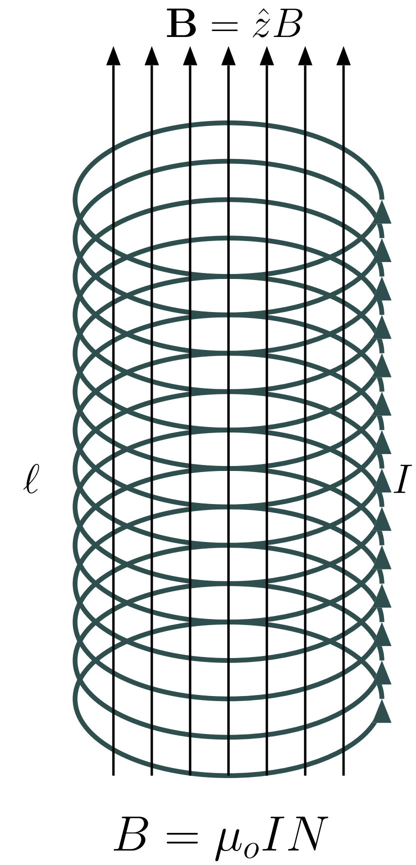
We continue with typical inductance calculations.

Inductance of long solenoid: Consider a long solenoid with length ℓ, cross-sectional area A, and a density of N loops per unit length as examined in Example 3 of Lecture 12 (see figure in the margin). As determined in Example 3, the magnetic flux density in the interior of the solenoid is

$$\mathbf{B} = \mu_o I N \hat{z}$$

while $n = N\ell$ is the number of turns of the solenoid. Thus, the inductance of the solenoid with $n = N\ell$ turns is

$$L = \frac{n\Psi}{I} = \frac{N\ell(\mu_o I N)A}{I} = N^2\mu_o A\ell.$$



- As we know from our circuit courses, an inductor L such as the solenoid coil considered above can be used to store energy. An inductor connected to an external circuit with a quasi-static current I develops a voltage drop $V = L\frac{dI}{dt}$ across its terminals[2] and absorbs power at an instantaneous rate

$$P = VI = L\frac{dI}{dt}I = \frac{d}{dt}(\frac{1}{2}LI^2),$$

implying a stored energy of

$$W = \frac{1}{2}LI^2 = \frac{1}{2}N^2\mu_o A\ell I^2 = \frac{|B_z|^2}{2\mu_o}A\ell = \frac{1}{2}\mu_o|H_z|^2 A\ell$$

in an inductor in a conducting state.

[2]Assuming a physical size much smaller than a wavelength $\lambda = c/f$ for the highest frequency in $I(t)$.

- Notice that the stored energy of the solenoid is

$$\frac{1}{2}\mu_o|H_z|^2 = \frac{1}{2}\mu_o\mathbf{H}\cdot\mathbf{H}$$

times its volume $A\ell$ occupied by the field $\mathbf{H}$ inside the solenoid. That suggests that

$$w = \frac{1}{2}\mu_o\mathbf{H}\cdot\mathbf{H}$$

can be interpreted as stored magnetostatic energy per unit volume in general.

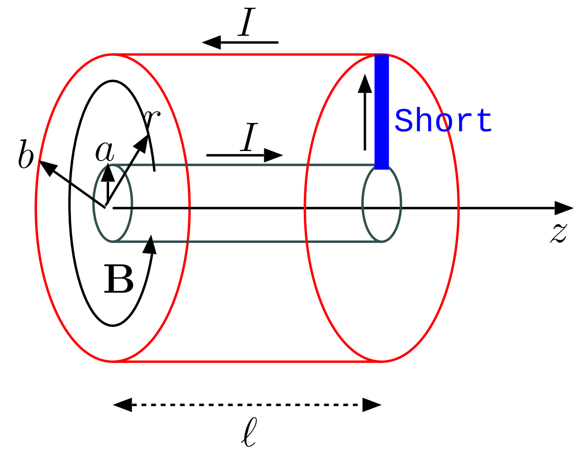
  - Also both inductance L and stored energies W and w would have μ replacing μ_o in material media with permeabilities

    $$\mu = (1+\chi_m)\mu_o$$

    and magnetic susceptibilities χ_m, in analogy with the concepts of permittivity $\epsilon = (1+\chi_e)\epsilon_o$ and electrical susceptibility χ_e.

    - Permeability and magnetic susceptibility notions will be examined in a future lecture.

Inductance of shorted coax: Consider a coaxial cable of some length ℓ which is "shorted" at one end (with a wire connecting the inner and outer conductors), so that a steady current I can flow on the inner conductor of radius a to return on the interior surface of the outer conductor at radius b after having circulated through the short. We will next determine the inductance L of such an inductor after first computing the magnetic flux density B_ϕ that will be produced by the inner conductor current I. In B_ϕ calculation we will assume $\ell \gg b$ so that an "infinite coax" approximation can be invoked.



Shorted coax circulates a current I linking a magnetic flux Ψ confined to a region bounded by the outer conductor of the coax.

- Expanding the integral form of Ampere's law

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_o I_C$$

  as

$$B_\phi 2\pi r = \mu_o I$$

  over a circular integration contour C of a radius $r > a$, we find that the magnetic flux density in the interior of the coax cable is

$$B_\phi = \frac{\mu_o I}{2\pi r}.$$

- Therefore, the magnetic flux linked by the closed current path I (see figure in the margin) is

$$\Psi = \int_S \mathbf{B} \cdot d\mathbf{S} = \ell \frac{\mu_o}{2\pi} I \int_a^b \frac{dr}{r} = \ell \frac{\mu_o}{2\pi} \ln \frac{b}{a} I.$$

Clearly, we have a linear relation $\Psi = LI$, with

$$L \equiv \frac{\ln \frac{b}{a}}{2\pi} \ell \mu_o,$$

which is the inductance of a shorted coax of a finite length ℓ.

- The inductance of the coax per unit length is

  $$\mathcal{L} = \frac{\ln \frac{b}{a}}{2\pi} \mu_o,$$

  which should be contrasted with capacitance per unit length

  $$\mathcal{C} = \frac{2\pi}{\ln \frac{b}{a}} \epsilon_o$$

  of the same coax configuration.

  Notice how $\mathcal{L}$ and $\mathcal{C}$ are proportional to ϵ_o and μ_o, respectively, having proportionality constants which are inverses of one another.

Inductance of shorted parallel plates: If a pair of parallel plates of areas $A = W\ell$ and separation d were shorted at one end, we would obtain effectively an inductor with a per length inductance

$$\mathcal{L} = \frac{d}{W}\mu_o$$

that accompanies per length capacitance

$$\mathcal{C} = \frac{W}{d}\epsilon_o$$

of the same parallel plate configuration. This follows from a generalization of our finding above that the proportionality constants of $\mathcal{L}$ and $\mathcal{C}$ are arithmetic inverses of one another.