14 Faraday's law and induced emf

Michael Faraday discovered (in 1831, less than 200 years ago) that a *changing* current in a wire loop *induces* current flows in nearby wires — today we describe this phenomenon as **electromagnetic induction**: the current change in the first loop causes the magnetic field produced by the current to change, and magnetic field change, in turn, is said to *induce*[1] (i.e., produce) electric fields which drive the currents in nearby wires.

- While static electric fields produced by static charge distributions are unconditionally curl-free, *time-varying electric fields* produced by current distributions with time-varying components are found to have, in accordance with Faraday's observations, non-zero curls specified by

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \text{Faraday's law}$$

  at all positions $\mathbf{r}$ in all reference frames of measurement. Using *Stoke's theorem*, the same constraint can also be expressed in *integral form* as

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} \qquad \text{Faraday's law}$$

  for all surfaces S bounded by all closed and *directed* paths C (with the direction of C, indicated by an arrow, and direction of vector $d\mathbf{S}$ related by right hand rule).

Definitions of E and B have not changed:

recall that

- **E is force per unit stationary charge**
- **B gives an additional force $\mathbf{v} \times \mathbf{B}$ per unit charge in motion with velocity v in the measurement frame.**

[1] Relativistic derivation of static **B** given in Lecture 12 can be extended to show that Coulomb interactions of charges in *time-varying* motions require a description in terms of time-varying **B** *and* **E** — see, e.g., *Am. J. Phys.*: Tessman, 34, 1048 (1966); Tessman and Finnel, **35**, 523 (1967); Kobe, **54**, 631 (1986). Thus, the *cause* of *induced* **E** is not really the time-varying **B**, but rather the time-varying current **J** that is also producing the variation in **B**.

- The right hand side of the integral form equation above includes the **flux of *rate of change of magnetic field* B** over surface S. If contour C bounding S is "fixed" (unchanging) in the measurement frame, then the equation can also be expressed as

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{S},$$

where the right hand side is now expressed in terms of the *rate of change* of **magnetic flux**

$$\Psi \equiv \int_S \mathbf{B} \cdot d\mathbf{S}$$

linking contour C over any surface S bounded by C.

- This modification (the exchange of the order of integration and time derivative on the right side) would *not* be permissible if path C were moving within the measurement frame or being deformed in time — but in such cases we could still express Faraday's integral form equation with $-\frac{d\Psi}{dt}$ on the right side, provided that we also modify the left side as in

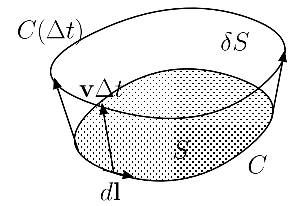
$$\oint_C (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l} = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{S}$$

where $\mathbf{v}$ denotes the velocity of motion or deformation of path C.

  - This is equivalent to the original equation, since, as shown in the margin,

$$\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} = \frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{S} + \oint_C \mathbf{v} \times \mathbf{B} \cdot d\mathbf{l}$$

    when C is changing continuously with velocities $\mathbf{v}$.



$\Psi(0) = \int_S \mathbf{B}(\mathbf{r}, 0) \cdot d\mathbf{S}$, and

$\Psi(\Delta t) = \int_S \mathbf{B}(\mathbf{r}, \Delta t) \cdot d\mathbf{S} + \int_{\delta S} \mathbf{B}(\mathbf{r}, \Delta t) \cdot d\mathbf{S}$.

Thus, $\frac{\Psi(\Delta t) - \Psi(0)}{\Delta t} =$

$$\int_S \frac{\mathbf{B}(\mathbf{r}, \Delta t) - \mathbf{B}(\mathbf{r}, 0)}{\Delta t} \cdot d\mathbf{S} + \int_{\delta S} \mathbf{B}(\mathbf{r}, \Delta t) \cdot \frac{d\mathbf{S}}{\Delta t}.$$

Hence in limit $\Delta t \to 0$

$$\frac{d\Psi}{dt} = \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} - \int_C \mathbf{v} \times \mathbf{B} \cdot d\mathbf{l},$$

since

$$\int_{\delta S} \mathbf{B}(\mathbf{r}, \Delta t) \cdot \frac{d\mathbf{S}}{\Delta t} = \int_C \mathbf{B}(\mathbf{r}, \Delta t) \cdot \frac{\Delta t \mathbf{v} \times d\mathbf{l}}{\Delta t} = -\int_C (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l}$$

because

$\mathbf{B} \cdot (\mathbf{v} \times d\mathbf{l}) = d\mathbf{l} \cdot (\mathbf{B} \times \mathbf{v})$,

both representing the volume of a parallelepiped formed by the vectors $d\mathbf{l}$, $\mathbf{v}$, and $\mathbf{B}$.

Note that velocity $\mathbf{v}$ does not have to be constant around contour C.

- A **physical interpretation** of the final equation

$$\oint_C (\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot d\mathbf{l} = \underbrace{-\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{S}}_{} \qquad \text{Integral form Faraday's Law}$$

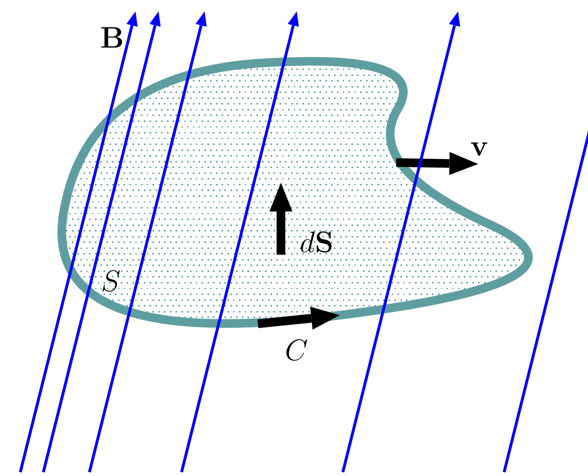
$$\mathcal{E} = -\frac{d\Psi}{dt} \qquad \text{Faraday's eqn.}$$

is as follows:

– the **circulation integral** on the left is the "voltage drop" once around the *directed closed path* C, representing the **work done per unit charge** (by the Lorentz force $\propto \mathbf{E}+\mathbf{v}\times\mathbf{B}$) taken a full circle around C, which was denoted by Michael Faraday with a symbol $\mathcal{E}$ and called the **emf** (short for *electro-motive force,* which is a bad name since $\mathcal{E}$ is *work,* and not *force,* per unit charge) for the closed path, equaling the decay rate $-\frac{d\Psi}{dt}$ of its *linked* **magnetic flux** Ψ (due to all sources of magnetic flux density $\mathbf{B}$ in the region).

– if/when path C is occupied by a conducting wire loop of some total conductance $G = \frac{1}{R}$, and a resistance $R = \frac{1}{G}$, a current $I = G\mathcal{E} = \frac{\mathcal{E}}{R}$ will flow around the loop in the circulation direction[2],

Magnetic field lines contributing to Ψ *form links* with path C (bounding S) like the links in an ordinary chain — hence, Ψ is said to be the *flux linking path* C.



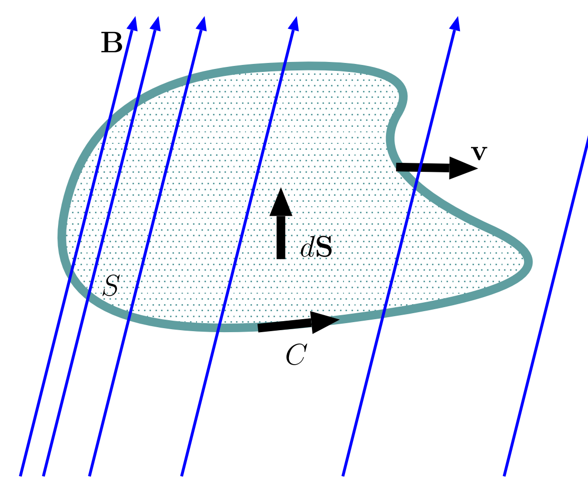
[2] $I = A\sigma|\mathbf{E}+\mathbf{v}\times\mathbf{B}|$ for a homogeneous wire loop with a conductivity σ and cross sectional area A. If the loop length is L, then the loop conductance is $G = \frac{A\sigma}{L}$ and therefore we find that $I = G\mathcal{E}$, as claimed, since $\mathcal{E} = \oint_C (\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot d\mathbf{l} = |\mathbf{E}+\mathbf{v}\times\mathbf{B}|L$ around a homogeneous loop.

driven by the non-zero field $\mathbf{E}+\mathbf{v}\times\mathbf{B}$ within the wire accounting for the non-zero $\mathcal{E}$ if/when $-\frac{d\Psi}{dt}$ is non-zero.

– in equivalent circuit models of conducting wire loops, Faraday's equation, re-written as $RI = -\frac{d\Psi}{dt}$, is effectively Kirchhoff's voltage law (KVL) applied to the loop, with RI on the left denoting the (sum of all) *voltage drops* in the direction of C, while $-\frac{d\Psi}{dt}$ on the right denoting a *voltage rise* also in the direction of C.

  ∘ note that the emf $\mathcal{E}$ describes both the *voltage drop* RI and *voltage rise* $-\frac{d\Psi}{dt}$ appearing in the circuit model for the conducting wire loop since $\mathcal{E} = RI$ and $\mathcal{E} = -\frac{d\Psi}{dt}$ are both true.

– in modern parlance (since Maxwell) the term emf and its symbol $\mathcal{E}$ are used to refer to and denote sources of energy, e.g., battery voltages and magnetic flux rate $-\frac{d\Psi}{dt}$ that drive currents $I = \frac{\mathcal{E}}{R}$ around closed circuits[3].



- If path C is *fixed* in the measurement frame, then $\mathbf{v} = 0$, and KVL for such a stationary loop reads as

$$\oint_C \mathbf{E}\cdot d\mathbf{l} = -\frac{d\Psi}{dt};$$

  – otherwise, that is if C is in motion, then

[3]see Saslow, *Am. J. Phys.*, **58**, 22 (2021), for a discussion of Maxwell's interpretation of emf and electrical energy production in batteries. Also see Scanlon et al., *Am. J. Phys.*, **37**, 689 (1969) for a discussion of $\mathcal{E} = \oint_C (\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot d\mathbf{l}$ vs $\mathcal{E} = -\frac{d\Psi}{dt}$.

$$\oint_C (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l} = -\frac{d\Psi}{dt}$$

because in that case force per unit charge *advected* with path C will be $\mathbf{E} + \mathbf{v} \times \mathbf{B}$ according to Lorentz force (note: any additional velocity $\mathbf{v}_q$ of a moving charge *along* C does not contribute because $(\mathbf{v}_q \times \mathbf{B}) \cdot d\mathbf{l} = 0$ if $d\mathbf{l}$ and $\mathbf{v}_q$ are parallel).

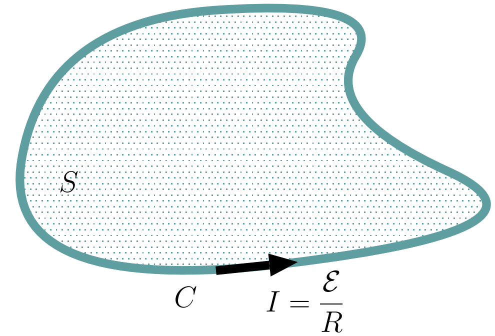
– In either case, if C is a physical conducting path with a total resistance R, then the emf $-\frac{d\Psi}{dt}$ drives a current

$$I = \frac{-\frac{d\Psi}{dt}}{R}$$

around C in the circulation direction (determined by $d\mathbf{l}$ and $d\mathbf{S}$ directions used in accordance with the right-hand-rule).

- The minus sign present in Faraday's equation, $\mathcal{E} = -\frac{d\Psi}{dt}$, assures that induced current I produces an induced magnetic field that *opposes* the flux change producing the emf — this fact is known as **Lenz's rule** and is in full accord with observations[4]./newpage

- According to Faraday's law it appears that magnetic flux variations $-\frac{d\Psi}{dt}$ can produce a non-zero emf independent of how the variations are produced — the possibilities are:



Think of EMF as the sum of all the "voltage rises" around the loop traversed in the direction of loop current I that needs to match the total "voltage drop" RI around the same loop traversed in the same direction.

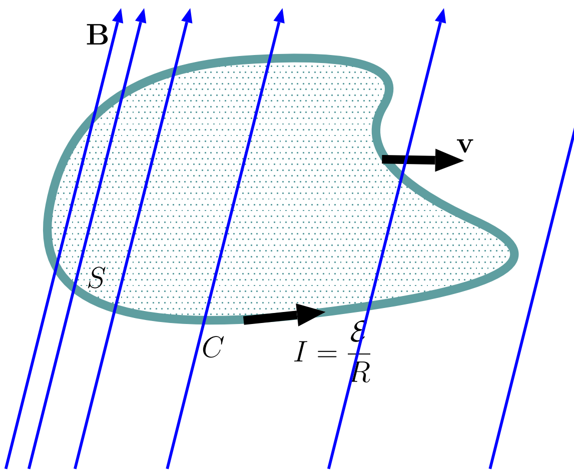
That way, KVL which states that

Sum of voltage rises = Sum of voltage drops,

is fulfilled.

[4]Faraday's law not having the minus sign (or in conflict with Lenz's rule) would be non-physical, as it would lead to unbounded growth of induced currents and fields (by aiding rather than opposing the flux change producing the emf).

1. Fixed C, but time-varying $\mathbf{B}$,
2. $\mathbf{B}$ =const. (in space and time), but time-varying C (rotating or changing size),
3. An inhomogeneous static $\mathbf{B} = \mathbf{B}(\mathbf{r})$ in the measurement frame *and* C in motion.



- Note that even in the absence of any electric field $\mathbf{E}$ in the measurement frame, a non-zero emf

$$\oint_C (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l} = -\frac{d\Psi}{dt}$$

can exist because of the motion of C through an *inhomogeneous* magnetic field (if the field is homogeneous then $\frac{d\Psi}{dt}$ will be zero, implying zero $\mathcal{E}$), which will of course appear as an emf

$$\oint_C \mathbf{E}' \cdot d\mathbf{l}' = -\frac{d\Psi'}{dt'}$$

for a second observer moving with C who sees a time varying electric field $\mathbf{E}' = \mathbf{v} \times \mathbf{B}$ in her own frame (in addition to the inhomogeneous but constant magnetic field $\mathbf{B}$ of the first frame appearing as a time-varying magnetic field $\mathbf{B}'$)[5].

[5] See Scanlon et. al., *Am. J. Phys.*, **37**, 698 (1969), for a discussion of $I' = \frac{\mathcal{E}'}{R}$ for rigid C with resistance R observed from different reference frames.

– Thus, having non-zero electric field circulations

$$\oint_C \mathbf{E}' \cdot d\mathbf{l}'$$

under time-varying magnetic field conditions appears to be quite *comprehensible* after all!

– Magnetic fields $\mathbf{B}$ in one frame will morph into electric fields $\mathbf{E}'$ in other frames because of (near) invariance of Lorentz force between reference frames.

– Moreover a morphed $\mathbf{E}'$ can even be non-conservative — i.e., non curl-free — when $\mathbf{B}$ is inhomogeneous in space (or time) as we have just seen.

Example 1: If

$$\mathbf{B} = B_o e^{-t/\tau} \hat{z},$$

what is the emf $\mathcal{E}$ taken over a stationary circular loop C of radius $r = 10$ m on $z = 0$ plane in counter-clockwise direction (looking down on $z = 0$ plane)? What is current I if the loop resistance is R?

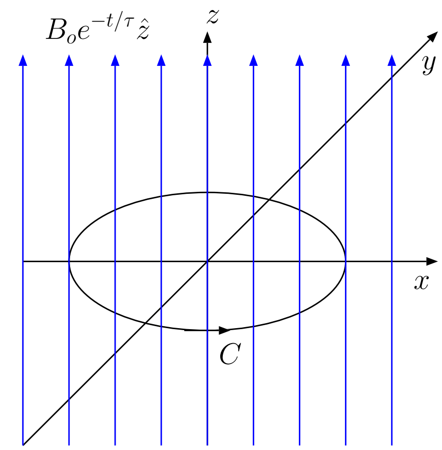
Solution: Since counter-clockwise circulation is requested we take $d\mathbf{S}$ pointing in $\hat{z}$ direction to be consistent with the right hand rule. We then have

$$\Psi = \int_S \mathbf{B} \cdot d\mathbf{S} = (B_o e^{-t/\tau} \hat{z}) \cdot (\pi 10^2 \hat{z}) = \pi 10^2 B_o e^{-t/\tau}$$

over the circular surface S. Thus, the emf

$$\mathcal{E} = -\frac{d\Psi}{dt} = \pi 10^2 \frac{B_o}{\tau} e^{-t/\tau}.$$

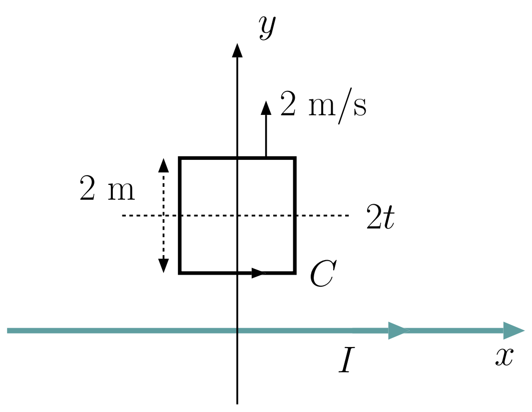
The loop current will be $I = \frac{\mathcal{E}}{R}$ in counter-clockwise direction of the computed circulation $\mathcal{E}$, which will be positive and counteract (i.e., strengthen) the weakening B_z.



Example 2: Consider the magnetic flux density

$$\mathbf{B} = \frac{\mu_o I}{2\pi r}\hat{\phi}$$

produced by current I flowing along the x axis. What is the emf $\mathcal{E}$ of a square loop C of area 4 m^2 moving on xy-plane with edges parallel to x- and y-axes, if its center is located at $y = 2t$ m as a function of time? Compute the emf $\mathcal{E}$ first as $-\frac{d\Psi}{dt}$ and then as $\oint_C (\mathbf{v}\times\mathbf{B})\cdot d\mathbf{l}$ to verify that the same values are obtained.



Solution: Given the described geometry, we have

$$\Psi(t) = \int_{-1}^{1} dx \int_{2t-1}^{2t+1} dy \frac{\mu_o I}{2\pi y} = \frac{\mu_o I}{\pi}\ln(\frac{2t+1}{2t-1}).$$

Thus, the emf $\mathcal{E}$ is

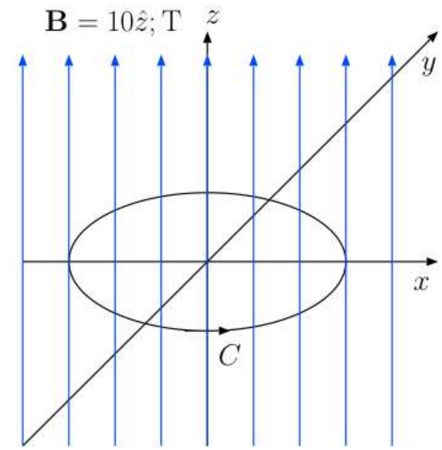
$$-\frac{d\Psi}{dt} = -\frac{\mu_o I}{\pi}(\frac{2t-1}{2t+1})\frac{\partial}{\partial t}(\frac{2t+1}{2t-1}) = \frac{\mu_o I}{\pi}\frac{4}{(2t+1)(2t-1)} = \frac{\mu_o I}{\pi(t^2-\frac{1}{4})}.$$

Alternatively, since $\mathbf{v} = 2\hat{y}$ m/s, and $\mathbf{v}\times\mathbf{B} = 2\frac{\mu_o I}{2\pi r}\hat{x}$, we find, using $d\mathbf{l} = \pm\hat{x}dx$ and $\pm\hat{y}dy$ in turns,

$$\mathcal{E} = \oint_C (\mathbf{v}\times\mathbf{B})\cdot d\mathbf{l} = 2\frac{\mu_o I}{2\pi(2t-1)}2 - 2\frac{\mu_o I}{2\pi(2t+1)}2 = \frac{\mu_o I}{\pi(t^2-\frac{1}{4})}$$

in consistency with the above result.

Example 3: A conducting loop of a radius $r = 0.1$ m (see figure in the margin) is being rotated about the x axis with frequency of $f = \frac{\omega}{2\pi} = 60$ Hz in a region with a DC magnetic field of $\mathbf{B} = 10\hat{z}$ T. Determine the induced current in the loop if the loop resistance is $12\,\Omega$.



Solution: The maximum value of the magnetic flux linking the loop should be

$$\Psi_o = \pi(0.1)^2 10 = 0.1\pi\,\mathrm{Wb}.$$

The time-varying flux linking the rotating loop is therefore

$$\Psi(t) = \Psi_o \cos(\omega t) = 0.1\pi\cos(120\pi t).$$

The corresponding emf is

$$\mathcal{E} = -\frac{d\Psi}{dt} = (120\pi)0.1\pi\sin(120\pi t).$$

Therefore, the induced current around the loop must be

$$I = \frac{\mathcal{E}}{R} = \frac{12\pi^2\sin(120\pi t)}{12} = \pi^2\sin(120\pi t)\,\mathrm{A}.$$

Example 4: A conducting bar of resistance $R_1 = 1\,\Omega$ ohms is moved in the x-direction with a velocity $\mathbf{v} = 3\hat{x}$ m/s on a pair of perfect conducting ($R = 0$) stationary rails 2 m apart terminated with a load resistance R_2 at $x = 0$, all constituting a rectangular contour C to be taken counterclockwise. A constant magnetic field of $\mathbf{B} = 1\hat{y}$ T is linked through contour C such that the flux $\Psi = -1 \times 2 \times 3t$ and the emf $\mathcal{E} = -d\Psi/dt = 6$ V. Hence, Faraday's law demands that

$$\oint_C (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l} = \int_b^t (\mathbf{E} + \mathbf{v} \times \mathbf{B})_1 \cdot d\mathbf{l} + \int_t^b (\mathbf{E})_2 \cdot d\mathbf{l} = 6$$

where the two integrals (with b and t referring to bottom and top rail contact points) correspond to voltage drops across resistors R_1 and R_2, respectively. But since

$$\int_b^t (\mathbf{v} \times \mathbf{B})_1 \cdot d\mathbf{l} = 3 \times 1 \times 2 = 6,$$

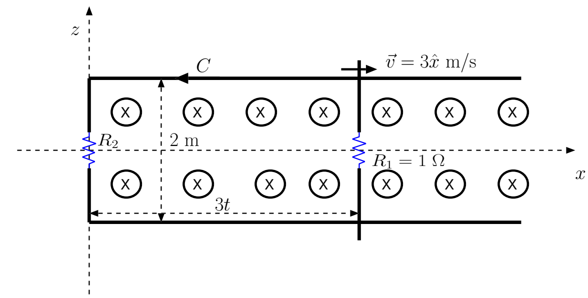
it follows that

$$\int_b^t (\mathbf{E})_1 \cdot d\mathbf{l} + \int_t^b (\mathbf{E})_2 \cdot d\mathbf{l} = 0 \;\Rightarrow\; E_{z1} - E_{z2} = 0 \;\Rightarrow\; E_{z2} = E_{z1},$$

i.e., identical static fields within the moving and stationary bars across the perfect conducting rails. This may be a surprising claim/result — let's give two examples to illustrate how this happens:

1. Let $R_2 = 2\,\Omega$ ohms. Then $I = 6/3 = 2$ A. It follows that voltage drops $(\mathbf{E} + \mathbf{v} \times \mathbf{B})_1 \cdot 2\hat{z} = 2$ V across R_1 and $(\mathbf{E})_2 \cdot (-2\hat{z}) = 4$ V across R_2, yielding $E_{z1} = E_{z2} = -2$ V/m.

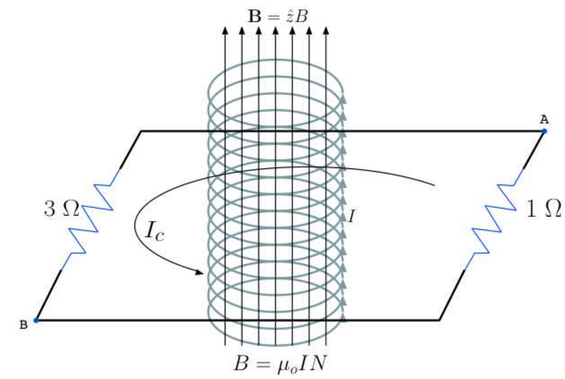
2. Let $R_2 = \infty$ — open ckt load to the moving conductor. Then $I = 6/\infty = 0$ A. It follows that $(\mathbf{E} + \mathbf{v} \times \mathbf{B})_1 \cdot 2\hat{z} = 0$ V across R_1 and $(\mathbf{E})_2 \cdot (-2\hat{z}) = 6$ V across R_2, yielding $E_{z1} = E_{z2} = -3$ V/m. Note that in this case the entire emf appears across the open termination (gap in the loop C and the emf $\int_b^t (\mathbf{E} + \mathbf{v} \times \mathbf{B})_1 \cdot d\mathbf{l} = 0$ across resistor R_1).



Moving bar in the presence of a constant magnetic field produces an emf and electric fields in the lab frame that drive a loop current I.

Example 4 illustrates how the $\oint \mathbf{E} \cdot d\mathbf{l}$ part of emf $\oint (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l}$ caused by a motion $\mathbf{v} = 3\hat{x}$ m/s is zero (with non-zero static E_z components)!!

Example 5: An infinite solenoid producing a *constant* $-\frac{d\Psi}{dt} = 8$ V, passes through small a loop consisting of a 1 Ω resistor on the right and a 3 Ω resistor on the left, connected in series — see margin plot. What is the current I_c through this resistor loop, and what voltages would be measured (by a voltmeter) across the individual resistors?



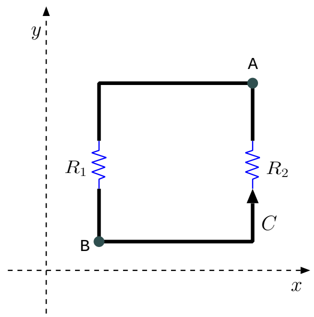
Solution: The magnetic flux produced by the solenoid will be confined to its interior as long as dI/dt (and thus $d\Psi/dt$, as specified) is constant and emf $\mathcal{E} = -d\Psi/dt$ is non-time varying (see below). In that case, with constant emf $\mathcal{E} = -\frac{d\Psi}{dt} = 8$ V of the encircling resistor loop in the setup, the loop current I_c is the ratio of $\mathcal{E}$ and the total loop resistance 4 Ω, i.e., $I_c = \frac{\mathcal{E}}{R} = 2$ A. Consequently, 1 and 3 Ω resistors will develop 2 and 6 V drops, respectively, in the direction of the 2A current!! Note that:

- the loop has no battery to support this current flow — it has instead been excited "inductively".
- with *constant* dI/dt, there is *zero* magnetic field at the locations of the loop wire and resistors (static $\mathbf{E}$ in the solenoid exterior is *curl-free*!) — thus, the emf of the loop is not being produced by a time varying local magnetic field; it is rather a consequence of the time-varying current $I(t)$ in the solenoid (which is also responsible for time-varying Ψ), with the relation $\mathcal{E} = -d\Psi/dt$ being "incidental"!
- what a voltmeter measures across the resistors — whether 2 or 6 V — depends on whether its probes contacting points A and B are placed to the right or to the left of the solenoid!! That's because the field $\mathbf{E}$ produced by the time-varying current $I(t)$ is no longer conservative across the system and consequently the line integral of $\mathbf{E}$ is path dependent — we have to be more careful about what we mean by *voltage* in these new situations!

Transformers which operate based on an inductive coupling principle, and electric dynamos (and motors) which produce motion induced emfs (and rotating coils) are studied in depth in power courses starting with ECE 330.

Example 6: Consider a square conducting loop of 1 m^2 cross sectional area bordered by $R_1 = 2\,\Omega$ and $R_2 = 1\,\Omega$ resistors as shown in the margin. The loop is linked with a magnetic flux Ψ due to time varying magnetic field described as $\mathbf{B} = (12 - 3t)\hat{z}$ T.

- Hence, $\Psi = 12 - 3t$ Wb and the emf $\mathcal{E} = -d\Psi/dt = 3$ V.
- Loop current $I = 3\text{V}/3\Omega = 1$ A in the circulation direction.
- Voltage drop $V_1 = 2$ V across R_1 from point A to point B.
- Voltage drop $V_2 = -1$ V across R_2 from point A to point B.
- A voltmeter connected from A (positive lead) to B will read 2 V if and only if its leads form a path identical to the path defined by R_1 (from A to B).
- A voltmeter connected from A (positive lead) to B will read -1 V if and only if its leads form a path identical to the path defined by R_2.
- A voltmeter connected from A (positive lead) to B will read 0.5 V if its leads form a diagonal path from A to B.
  - To see this, notice that Faraday's law applied for the triangular loop including the voltmeter and R_2 would have an emf of 1.5 V equaling the sum of voltmeter reading V_R and 1 V drop across resistor R_2.



In the presence of time varying magnetic flux, voltage of a path P, defined as $\int_P (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l}$, will in general be path dependent!

A voltmeter reads and displays the voltage of its own path constituted by the placement of its own probe wires contacting the measurement nodes A and B.