9 Static fields in dielectric media

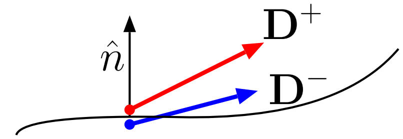
- Summarizing important results from last lecture:

  – within a dielectric medium, displacement

  $$\mathbf{D} = \epsilon\mathbf{E} = \epsilon_o\mathbf{E} + \mathbf{P},$$

  and if the permittivity $\epsilon = \epsilon_r\epsilon_o$ is known, $\mathbf{D}$ and $\mathbf{E}$ can be calculated from free surface charge ρ_s or volume charge ρ in the region without resorting to $\mathbf{P}$.

  – on surfaces separating perfect dielectrics, $\hat{n}\cdot(\mathbf{D}^+ - \mathbf{D}^-) = 0$ typically, while $\hat{n}\cdot\mathbf{D}^+ = \rho_s$ on a conductor-dielectric interface (with $\hat{n}$ pointing from the conductor toward the dielectric).



  – Gauss's law $\nabla\cdot\mathbf{D} = \rho$ (and its integral counterpart) includes only the free charge density on its right side, which is typically zero in many practical problems.

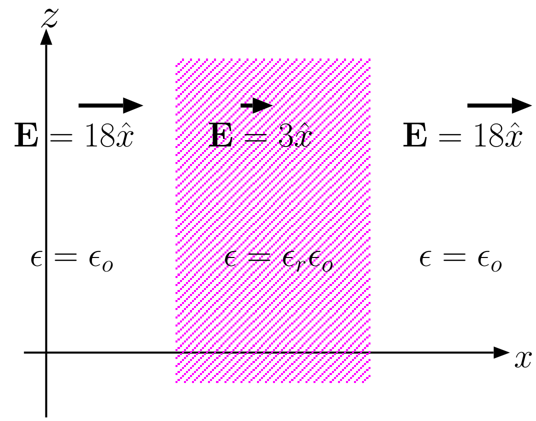
  – once $\mathbf{D}$ and $\mathbf{E}$ have been calculated (typically using the boundary condition equations), polarization $\mathbf{P}$ can be obtained as

  $$\mathbf{P} = \mathbf{D} - \epsilon_o\mathbf{E}$$

  if needed.

These rules will be used in the examples in this section.

Example 1: A perfect dielectric slab having a finite thickness W in the x direction is surrounded by free space and has a constant electric field $\mathbf{E} = 18\hat{x}$ V/m in its exterior. Induced polarization of bound charges inside dielectric reduces the electric field strength inside the slab from $18\hat{x}$ V/m to $\mathbf{E} = 3\hat{x}$ V/m. What are the displacement field $\mathbf{D}$ and polarization $\mathbf{P}$ outside and inside the slab, and what are the dielectric constant ϵ_r and electric susceptibility χ_e of the slab?



Solution: Displacement field outside the slab, where $\epsilon = \epsilon_o$, must be

$$\mathbf{D} = \epsilon_o\mathbf{E} = \hat{x}18\epsilon_o \frac{\mathrm{C}}{\mathrm{m}^2}.$$

The outside polarization $\mathbf{P}$ is of course zero. Boundary conditions at the interface of the slab with free space require the continuity of normal component of $\mathbf{D}$ and tangential component of $\mathbf{E}$ — both of these conditions would be satisfied if we were to take $\mathbf{D} = \hat{x}18\epsilon_o$ C/m^2 also within the dielectric slab. Thus, with $\mathbf{E} = 3\hat{x}$ V/m inside the slab, the condition $\mathbf{D} = \epsilon_{slab}\mathbf{E}$ within the slab requires that

$$\epsilon_{slab} = 6\epsilon_o.$$

Consequently, the dielectric constant of the slab is

$$\epsilon_r = 1 + \chi_e = \frac{\epsilon_{slab}}{\epsilon_o} = 6$$

and its electric susceptibility is

$$\chi_e = \epsilon_r - 1 = 5.$$

Finally, since $\mathbf{D} = \epsilon_o\mathbf{E} + \mathbf{P}$ in general, polarization $\mathbf{P}$ inside the slab is

$$\mathbf{P} = \mathbf{D} - \epsilon_o\mathbf{E} = \hat{x}18\epsilon_o - \epsilon_o 3\hat{x} = \hat{x}15\epsilon_o \frac{\mathrm{C}}{\mathrm{m}^2}.$$

- Our revised definition of displacement $\mathbf{D} = \epsilon\mathbf{E}$, where $\epsilon = \epsilon_r\epsilon_o$, implies, when combined with $\mathbf{E} = -\nabla V$ and $\nabla \cdot \mathbf{D} = \rho$, a revised form of Poisson's equation

$$\nabla^2 V = -\frac{\rho}{\epsilon},$$

  - provided that dielectric constant ϵ_r is independent of position so that $\nabla \cdot \mathbf{D} = \nabla \cdot (\epsilon\mathbf{E}) = \epsilon\nabla \cdot \mathbf{E}$ is a valid intermediate step in the derivation of Poisson's equation.
  - Under the same condition Laplace's equation $\nabla^2 V = 0$ also remains valid.
  - Dielectrics where ϵ_r is independent of position are said to be **homogeneous**.
    - In **inhomogeneous** dielectrics where ϵ varies with position neither equation is valid, and one has to resort to the full form of Gauss's law in field and potential calculations.

  In other words, don't use Laplace's/Poisson's equations in inhomogeneous media.

  In the next example we have two homogeneous slabs side-by-side making up an inhomogeneous configuration. In that case we can use Laplace/Poisson within the slabs one at a time and then match the results at the boundary using boundary condition equations as shown.

Example 2: A pair of infinite conducting plates at $z = 0$ and $z = 2$ m carry equal and opposite surface charge densities of $-2\epsilon_o$ C/m^2 and $2\epsilon_o$ C/m^2, respectively. Determine $V(2)$ if $V(0) = 0$ and regions $0 < z < 1$ m and $1 < z < 2$ m are occupied by perfect dielectrics with permittivities of ϵ_o and $2\epsilon_o$, respectively.

Solution: Given that $V(0) = 0$, we assume $V(z) = Az$, for some constant A in the homogeneous region $0 < z < 1$ m, since $V(z) = Az$ satisfies the Laplace's equation as well as the boundary condition at $z = 0$.

This gives $V(1) = A$ at $z = 1$ m, which then implies that we can take $V(z) = A + B(z-1)$ for the second homogeneous region $1 < z < 2$ m having a different permittivity than the region below.

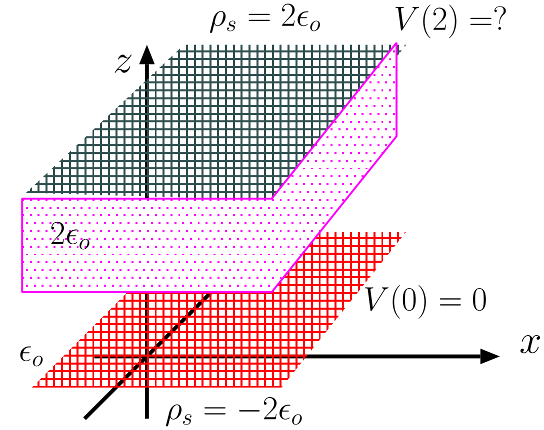
To determine the constants A and B, we will make use of boundary conditions at $z = 0$ and $z = 1$ m interfaces:

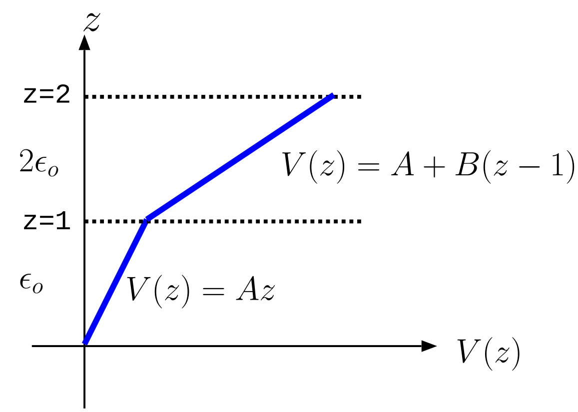
- In the region $0 < z < 1$ m, the electric field $\mathbf{E} = -\nabla(Az) = -A\hat{z}$, and, therefore displacement $\mathbf{D} = \epsilon_1\mathbf{E} = -\epsilon_o A\hat{z}$. Hence, the pertinent boundary condition $\hat{z}\cdot\mathbf{D}(0) = \rho_s$ yields

$$\hat{z}\cdot\mathbf{D}(0) = -\epsilon_o A = -2\epsilon_o \quad \Rightarrow \quad A = 2.$$

- Just below $z = 1$ m the displacement is $\mathbf{D}(1^-) = -\epsilon_o A\hat{z} = -2\epsilon_o\hat{z}$ as we found out above. Above $z = 1$ m, the electric field is $\mathbf{E} = -\nabla(A + B(z-1)) = -B\hat{z}$, and, therefore, $\mathbf{D}(1^+) = -2\epsilon_o B\hat{z}$ just above $z = 1$ m. Hence, the pertinent boundary condition $\hat{z}\cdot(\mathbf{D}(1^+) - \mathbf{D}(1^-)) = 0$ yields

$$\hat{z}\cdot(-2\epsilon_o B\hat{z} - (-2\epsilon_o\hat{z})) = -2\epsilon_o B + 2\epsilon_o = 0 \quad \Rightarrow \quad B = 1.$$



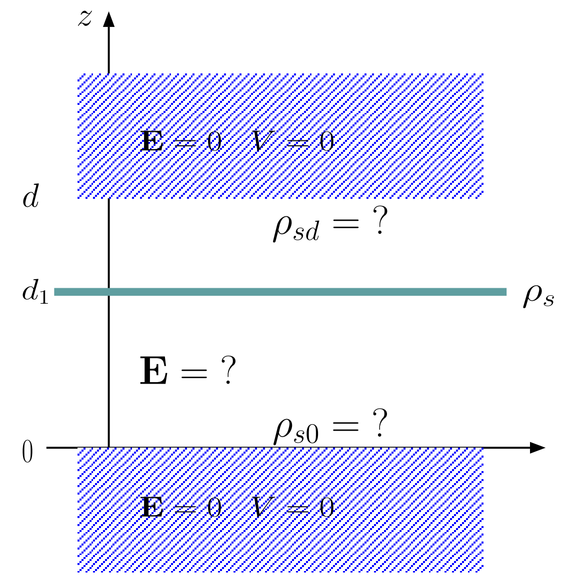


Based on above calculations of constants A and B, the potential solution for the region is

$$V(z) = \begin{cases} 2z\,\mathrm{V}, & 0 < z < 1 \\ 2 + (z-1)\,\mathrm{V}, & 1 < z < 2. \end{cases}$$

It follows that $V(2) = 3$ V.

Note that electric fields $-2\hat{z}$ V/m and $-\hat{z}$ V/m in the bottom and top layers point from high to low potential regions. Electric field $\mathbf{E}$ is discontinuous at the boundary at $z = 1$ m while displacement $\mathbf{D}$ is continuous — the continuity of normally directed $\mathbf{D}$ is demanded by boundary condition equations in the absence of surface charge.



If ρ_s in Example 3 is a slowly-varying function of time, then slowly varying $\mathbf{E}$, ρ_{s0}, and ρ_{sd} calculated with instantaneous values of ρ_s would constitute *quasi-static solutions* which are valid so long as $d \ll c/f$, with f the highest frequency in $\rho_s(t)$.

Example 3: A pair of infinite conducting plates at $z = 0$ and $z = d$ are grounded and have equal potentials, say, $V = 0$. The region $0 < z < d$ is occupied by free space (i.e., $\epsilon = \epsilon_o$) except that an infinite charge sheet with a static surface charge density ρ_s is located at $z = d_1 < d$. Determine (a) the electrostatic field $\mathbf{E}(z)$ in regions $0 < z < d_1$ and $d_1 < z < d$, and (b) the surface charge densities ρ_{s0} and ρ_{sd} at $z = 0$ and $z = d$ on conductor surfaces if $d_1 = d/2$.

Solution: (a) Laplace's equation for the given geometry requires a linear (in z) potential solution in regions $0 < z < d_1$ and $d_1 < z < d$. Since electrostatic $\mathbf{E} = -\nabla V$, we can therefore represent the electric field in these regions as

$$\mathbf{E} = \begin{cases} -\hat{z}V_o/d_1, & 0 < z < d_1 \\ +\hat{z}V_o/d_2, & d_1 < z < d \end{cases}$$

where $V_o \equiv V(d_1)$ and $d_2 \equiv d - d_1$. Hence,

$$\mathbf{D} = \epsilon_o \mathbf{E} = \begin{cases} -\hat{z}\epsilon_o V_o/d_1, & 0 < z < d_1 \\ +\hat{z}\epsilon_o V_o/d_2, & d_1 < z < d \end{cases},$$

and Maxwell's boundary condition equation applied on $z = d_1$ surface is

$$\hat{z} \cdot (\mathbf{D}(d_1^+) - \mathbf{D}(d_1^-)) = \rho_s \quad \Rightarrow \quad \epsilon_o V_o \left(\frac{1}{d_2} + \frac{1}{d_1} \right) = \rho_s.$$

Thus

$$V_o = \frac{\rho_s}{\epsilon_o} \left(\frac{1}{d_2} + \frac{1}{d_1} \right)^{-1} = \frac{\rho_s}{\epsilon_o} \frac{d_1 d_2}{d_1 + d_2} = \frac{\rho_s}{\epsilon_o} \frac{d_1 d_2}{d}.$$

Substituting V_o back into the expression for $\mathbf{E}$, we have

$$\mathbf{E} = \begin{cases} -\hat{z}\frac{\rho_s}{\epsilon_o}\frac{d_2}{d}, & 0 < z < d_1 \\ +\hat{z}\frac{\rho_s}{\epsilon_o}\frac{d_1}{d}, & d_1 < z < d. \end{cases}$$

(b) The surface charge at $z = 0$ can be found by evaluating $\hat{z} \cdot \mathbf{D} = \hat{z} \cdot \epsilon_o \mathbf{E}$ at $z = 0$. Hence,

$$\rho_{s0} = \hat{z} \cdot \epsilon_o \mathbf{E}(0) = -\frac{d_2}{d}\rho_s \xrightarrow{d_1 = d/2} -\frac{\rho_s}{2}.$$

Likewise,

$$\rho_{sd} = -\hat{z} \cdot \epsilon_o \mathbf{E}(d) = -\frac{d_1}{d}\rho_s \xrightarrow{d_1 = d/2} -\frac{\rho_s}{2}.$$

Example 4: Between a pair of infinite conducting plates at $z = 0$ and $z = 2$ m, the medium is a perfect dielectric with an **inhomogeneous** permittivity of

$$\epsilon(z) = \frac{4\epsilon_o}{4 - z}.$$

Determine the electric potential $V(2)$ on the top plate if $V(0) = 0$ and the surface charge density is $\rho_s = 2\epsilon_o$ C/m^2 on the bottom plate at $z = 0$. Note that Laplace's equation cannot be used in this problem since the medium is inhomogeneous.

Solution: Consider Gauss's law

$$\nabla \cdot (\epsilon \mathbf{E}) = \rho$$

with $\rho = 0$ in the region $0 < z < 2$ m. Assuming that $\mathbf{E} = \hat{z}E_z(z)$, because the geometry is invariant in x and y, we have

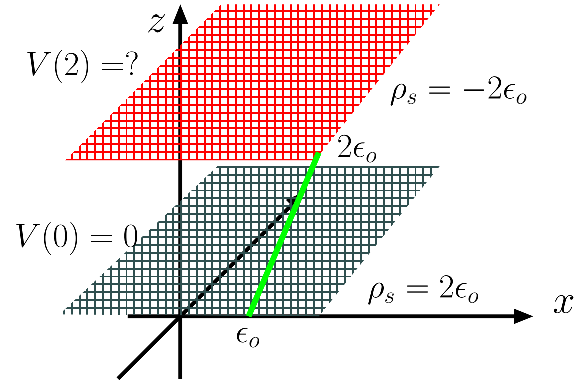
$$\nabla \cdot (\epsilon \mathbf{E}) = 0 \;\Rightarrow\; \frac{\partial}{\partial z}(\epsilon E_z) = 0 \;\Rightarrow\; \epsilon E_z = \text{constant}.$$

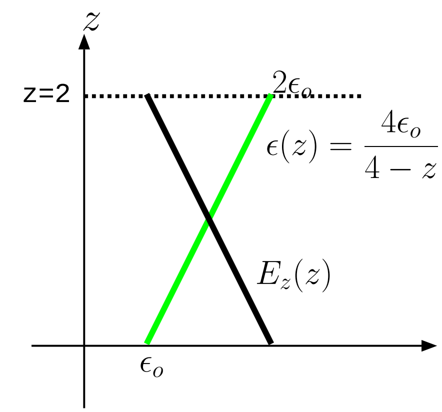
Thus the product ϵE_z is invariant with respect to coordinate z, which implies that

$$\epsilon(z)E_z(z) = \epsilon(0)E_z(0) \;\Rightarrow\; E_z(z) = \frac{\epsilon(0)}{\epsilon(z)}E_z(0) = E_z(0)(1 - \frac{z}{4})$$

after substituting for $\epsilon(z)$. To identify $E_z(0)$, we apply the bottom boundary condition $\hat{z} \cdot \mathbf{D}(0) = \rho_s$, and obtain

$$D_z(0) = \epsilon(0)E_z(0) = 2\epsilon_o \;\Rightarrow\; E_z(0) = \frac{2\epsilon_o}{\epsilon(0)} = 2\,\frac{\text{V}}{\text{m}}.$$





To determine $V(2)$, we integrate $\mathbf{E} = \hat{z}2(1-\frac{z}{4})$ V/m from top to bottom plate (grounded), obtaining

$$\begin{aligned} V(2) &= \int_{z=2}^{0} \mathbf{E}\cdot d\mathbf{l} = \int_{z=2}^{0} 2(1-\frac{z}{4})dz \\ &= 2(z-\frac{z^2}{8})|_2^0 = -2(2-\frac{4}{8}) = -2\cdot\frac{3}{2} = -3\,\text{V}. \end{aligned}$$