8 Conductors, dielectrics, and polarization

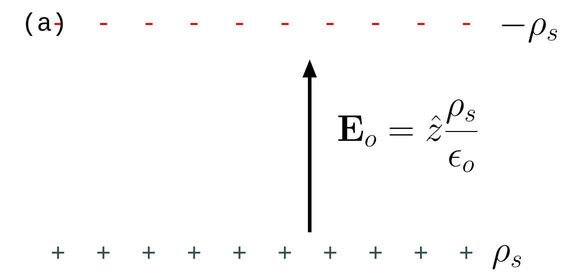
So far in this course we have examined static field configurations of charge distributions assumed to be fixed in free space in the *absence* of nearby materials (solid, liquid, or gas) composed of neutral atoms and molecules.

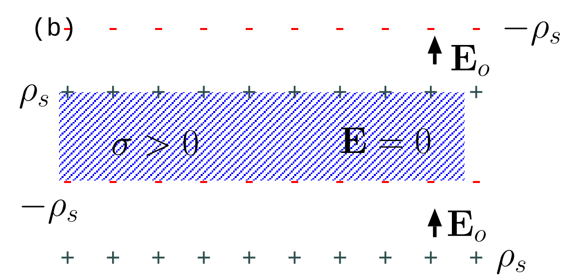
In the *presence* of material bodies composed of large number of charge-neutral atoms (in fluid or solid states) static charge distributions giving rise to electrostatic fields can be typically[1] found:

1. On exterior surfaces of *conductors* in "steady-state",

2. In crystal lattices occupied by *ionized* atoms, as in depletion regions of semiconductor junctions in diodes and transistors.

In this lecture we will examine these configurations and response of materials to applied electric fields.





A conducting slab inserted into a region with field E_o (as shown in b)develops surface charge which cancels out E_o within the slab.

E_o relates to surface charge as dictated by Gauss's law and superposition principle.

Conductivity and static charges on conductor surfaces:

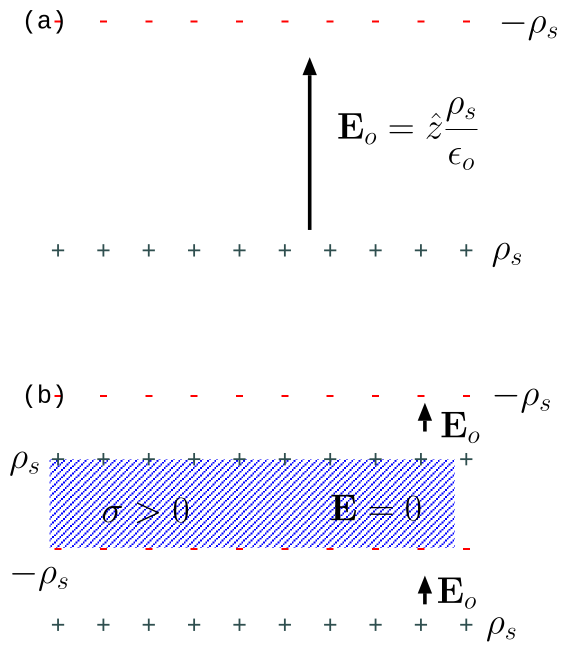
- **Conductivity** σ is an *emergent property* of materials bodies containing free charge carriers (e.g., unbound electrons, ionized atoms or molecules) which relates the applied *electric field* $\mathbf{E}$ (V/m) to the *electrical current density* $\mathbf{J}$ (A/m^2) conducted in the material via a linear relation[2]

[1] More generally, materials containing charge carriers exhibiting divergence free flows will also exhibit static charge distributions.

[2] Linear behavior is possible provided charge carriers suffer occasional collisions within the medium.

$$\mathbf{J} = \sigma \mathbf{E}. \quad \text{(Ohm's Law)}$$

- Simple physics-based models for σ will be discussed later in Lecture 11. For now it is sufficient to note that:
  - $\sigma \to \infty$ corresponds to a *perfect electrical conductor*[3] (PEC) for which it is necessary that $\mathbf{E} = 0$ (in analogy with $V = 0$ across a short circuit element) independent of $\mathbf{J}$.
  - $\sigma \to 0$ corresponds to a *perfect insulator* for which it is necessary that $\mathbf{J} = 0$ (in analogy with $I = 0$ through an open circuit element) independent of $\mathbf{E}$.
- While (macroscopic) $\mathbf{E} = 0$ in PEC's *unconditionally*, a conductor with a *finite* σ (e.g., copper or sea water) will also have $\mathbf{E} = 0$ in "steady-state" after the decay of transient currents $\mathbf{J}$ that may be initiated within the conductor after applying an external electric field $\mathbf{E}_o$ (see margin).
  - The reason is, mobile free charges (e.g., electrons in metallic conductors) within the conductor will be pulled or pushed by the applied field $\mathbf{E}_o$ to pile up on exterior surfaces of the conductor



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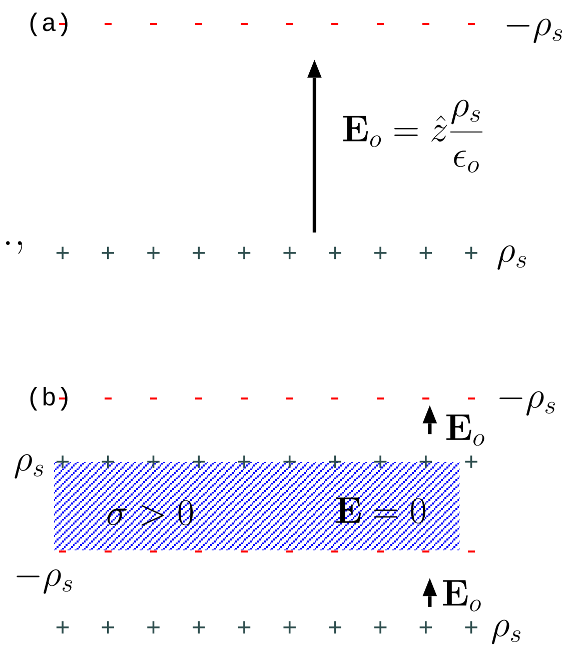
[3]PEC is an "idealization" that has no real counterpart, even though it is convenient to treat high conductivity materials such as copper as PEC in certain approximate models and calculations. For "superconducting materials" $\sigma \to \infty$ only in the low frequency limit.

until a surface charge density ρ_s that is generated produces a secondary field $-\mathbf{E}_o$ that exactly cancels out the applied $\mathbf{E}_o$ within the interior of the conductor.

- $\mathbf{E} = 0$ in the interior at steady-state implies that potential V =const., as well as $\rho = \nabla \cdot \mathbf{D} = \nabla \cdot \epsilon_o \mathbf{E} = 0$.
- Surface charge density ρ_s and the exterior field on a conductor surface will satisfy the boundary condition equations

$$\hat{n} \cdot \mathbf{D} = \rho_s \quad \text{and} \quad \hat{n} \times \mathbf{E} = 0,$$

with $\hat{n}$ denoting the outward unit normal.



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E_o relates to surface charge as dictated by Gauss's law and superposition principle.

- The transient "time-constant" τ for the *decay* of charge density ρ (and hence $\mathbf{E}$, as claimed above) in a homogeneous[4] conductor (constant σ) can be obtained using the *continuity equation*

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$

representing the mathematical statement of **charge conservation** (derived in Lecture 16). Using $\mathbf{J} = \sigma \mathbf{E}$ and $\nabla \cdot \mathbf{E} = \rho/\epsilon_o$, we have

$$\nabla \cdot \mathbf{J} = \sigma \nabla \cdot \mathbf{E} = \frac{\sigma}{\epsilon_o}\rho$$

[4]See Fisher and Varney, *Am. J. Phys.*, **44**, 464 (1976), for a discussion of contact potential between different metals.

above, from which it follows that

$$\frac{\partial \rho}{\partial t} + \frac{\sigma}{\epsilon_o}\rho = 0 \text{ with a damped solution } \rho(t) = \rho(0)e^{-\frac{\sigma}{\epsilon_o}t}.$$

The decay time-constant

$$\tau = \frac{\epsilon_o}{\sigma}$$

is typically very short ($\sim 10^{-18}$ s) in metallic conductors, which is why such conductors are usually considered to be in steady-state (and have zero interior fields).

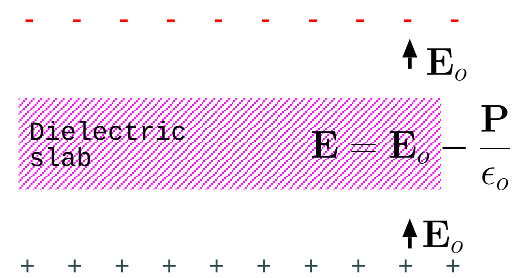
- **As a consequence:** in electrostatic[5] problems conducting volumes of materials (e.g., chunks of copper) can be treated as *equipotentials* having zero internal fields and finite surface charge densities $\rho_s = \hat{n} \cdot \mathbf{D}$ expressed in terms of external fields $\mathbf{D}$ normal to the surface.

[5]Also applicable *quasi-statically* when externally applied fields $\mathbf{E}_o(t)$ change slowly with time-constants much longer than ϵ_o/σ. The way conductors are treated in high frequency *electromagnetic* problems will be described later on.

Dielectric materials and polarization:

- **Dielectric materials** consist of a large number of charge-neutral atoms or molecules and *ideally* contain no mobile charge carriers (i.e., $\sigma = 0$).

- Electric fields produced by charges located outside or within a dielectric material will **polarize** the dielectric — meaning that its constituent atoms or molecules will be "stretched out" to expose their internal or "bound" charges, electrons and protons — which will in turn cause the electric field inside the dielectric to become *weaker* than (but not zero, as in conductors) what the field would have been in the absence of polarization effect.



A dielectric slab inserted into a region with an initial field E_o will become polarized.

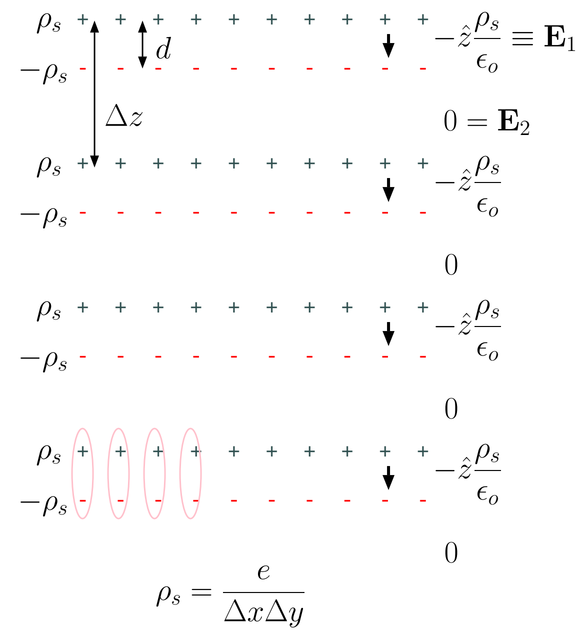
Inside the polarized dielectric the field will be weaker than E_o, but not reduced to zero as in a conductor.

We will next examine this polarization process and see how Gauss's law can be re-stated to facilitate field calculations in dielectric materials containing **bound charge** carriers, i.e., atomic/molecular electrons and protons which are not free to drift away from one another indefinitely (neglecting possible ionization events).

- Consider a static **free-charge** density $\rho(z) = \rho_f$ that would produce a macroscopic field $\mathbf{E}_o$ satisfying $\rho = \epsilon_o \nabla \cdot \mathbf{E}_o$ in free space, producing, instead, a field $\mathbf{E} = \hat{z} E_z$ inside a dielectric medium composed of an array of neutral atoms or molecules.

Our objective is to relate the field E to $\mathbf{E}_o$ and ρ_f, and find a way of calculating E when ρ_f is given.

- In the presence of an electric field $\mathbf{E} = \hat{z}E_z$ in the dielectric each neutral atom of the medium will be in a distorted (but not ripped apart) state forming a $\hat{z}$ oriented **electric dipole**, which can be visualized as a proton-electron pair with a small proton displacement d in z direction with respect to the electron.



  - Consider a regular array of such dipoles

$$\mathbf{p} \equiv ed\hat{z},$$

    with Δx, Δy, and Δz spacings between the dipoles (see margin), so that the volumetric dipole density is

$$N_d \equiv \frac{1}{\Delta x \Delta y \Delta z} \ \mathrm{m}^{-3},$$

    within the array, and, furthermore,

$$\rho_s = \frac{e}{\Delta x \Delta y} \ \frac{\mathrm{C}}{\mathrm{m}^2}$$

    is the magnitude of charge density of the adjacent proton and electron layers (see margin again) formed by arrays of adjacent dipoles displaced in z by intervals Δz.

  - Assuming that the array is infinite in extent in x and y directions, the proton and electron layers with surface charge densities $\pm\rho_s$ will produce interior electric fields

$$\mathbf{E}_1 = -\hat{z}\frac{\rho_s}{\epsilon_o} = -\hat{z}\frac{e/\epsilon_o}{\Delta x \Delta y}$$

(pointing in opposite direction to $\mathbf{E} = \hat{z}E_z$), and exterior fields

$$\mathbf{E}_2 = 0$$

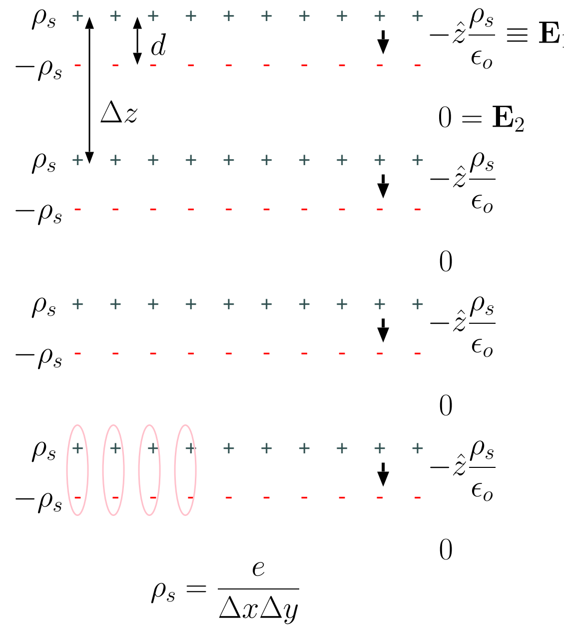
in between the dipole layers. Space averaged macroscopic electric field within the array (with a spatial weighting proportional to the size of regions with the fields $\mathbf{E}_1$ and $\mathbf{E}_2$) produced by the polarized dipoles will then be

$$\mathbf{E}_p = \mathbf{E}_1 \frac{d}{\Delta z} + \mathbf{E}_2 \frac{\Delta z - d}{\Delta z} = -\hat{z}\frac{ed/\epsilon_o}{\Delta x \Delta y \Delta z} = -\frac{N_d e d\hat{z}}{\epsilon_o} = -\frac{\mathbf{P}}{\epsilon_o},$$

where

$$\mathbf{P} \equiv N_d e d\hat{z} = N_d \mathbf{p}$$

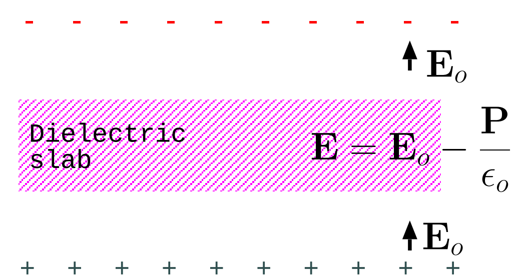
is, by definition, macroscopic **polarization field** of the dielectric, measured in units of C/m^2 (same units as a surface charge density).



– The total macroscopic field $\mathbf{E}$ in the dielectric is then the sum of field $\mathbf{E}_o$ produced by the free charge density ρ_f in the region and the macroscopic polarization field $\mathbf{E}_p = -\frac{\mathbf{P}}{\epsilon_o}$ produced by bound charge carriers of the neutral atoms and/or molecules of the dielectric, i.e.,

$$\mathbf{E} = \mathbf{E}_o - \frac{\mathbf{P}}{\epsilon_o},$$

a result that shows a "reduced field strength" $\mathbf{E}$ (compared to $\mathbf{E}_o$) inside the dielectric since $\mathbf{P}$ and $\mathbf{E}_o$ are collinear.



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- Let's re-arrange the expression for $\mathbf{E}$ from above as

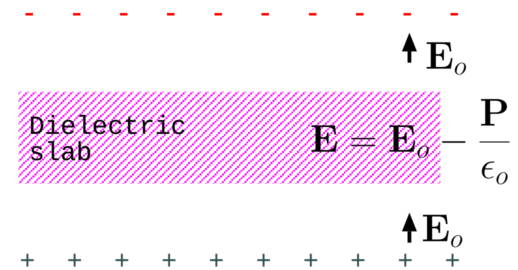
$$\epsilon_o \mathbf{E} + \mathbf{P} = \epsilon_o \mathbf{E}_o$$

after multiplying it with ϵ_o and moving $\mathbf{P}$ to the left. Now, the term on the right is $\epsilon_o \mathbf{E}_o = \mathbf{D}_o$ representing the displacement vector outside the dielectric slab[6], and if we were to "adopt" the left hand side expression as the "displacement vector" for the interior, i.e, take

$$\mathbf{D} = \epsilon_o \mathbf{E} + \mathbf{P}$$

in regions with non-zero $\mathbf{P}$, then we would see that

  - $\mathbf{D} = \mathbf{D}_o$, i.e., the displacement is the same inside and outside the slab, while electric fields $\mathbf{E}$ and $\mathbf{E}_o$ inside and outside differ by a non-zero $-\mathbf{P}/\epsilon_o$, and furthermore,
  - this generalized definition of electric displacement (a macroscopic field since $\mathbf{P}$ is macroscopic) is consistent with (by now familiar) $\mathbf{D} = \epsilon_o \mathbf{E}$ for free space since in free space $\mathbf{P} = 0$.



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- To express Gauss's law $\nabla \cdot (\epsilon_o \mathbf{E}) = \rho$ in a form applicable with our new revised $\mathbf{D} = \epsilon_o \mathbf{E} + \mathbf{P}$, we first note that Gauss's law (derived from

[6]This is true for the *infinite dielectric slab* geometry we are considering here where the external field $\mathbf{E}_o$ is not influenced by the polarized slab.
Outside a dielectric *sphere*, on the other hand, the external field will differ from the applied field $\mathbf{E}_o$ (see *Purcell*, Electricity and magnetism, 1965, Chapter 9) because of external fringing fields of finite sized bound charge layers of the dielectric sphere (giving rise to a non-zero external $\mathbf{E}_p$ in regions where $\mathbf{P} = 0$, and $\mathbf{E}_p = -\frac{\mathbf{P}}{3\epsilon_o}$ within the sphere, which in turn leads to $\mathbf{D} = \epsilon_o \mathbf{E} + \mathbf{P} = \epsilon_o \mathbf{E}_o + \frac{2}{3}\mathbf{P}$ within the sphere, differing from unperturbed displacement $\mathbf{D}_o = \epsilon_o \mathbf{E}_o$ to be seen far away from the sphere).

Coulomb's law via superposition over all microscopic charges) holds in any type of media so long as ρ is understood to be the total charge density $\rho = \rho_f + \rho_b$, a sum of charge densities ρ_f and ρ_b associated with free and bound charge carriers that could exist in the region. As such:

- outside dielectrics, Gauss's law is $\nabla \cdot (\epsilon_o \mathbf{E}) = \rho_f$, since $\rho_b = 0$ in that case, and this can be expressed as $\nabla \cdot \mathbf{D} = \rho_f$, with $\mathbf{D} = \epsilon_o \mathbf{E}$ as usual for free space where $\mathbf{P} = 0$;
- evaluating $\nabla \cdot \mathbf{D} = \rho_f$ with the macroscopic displacement field $\mathbf{D} = \epsilon_o \mathbf{E} + \mathbf{P}$ within a dielectric, we obtain $\nabla \cdot (\epsilon_o \mathbf{E}) = \rho_f - \nabla \cdot \mathbf{P}$, implying, in view of Gauss' law $\nabla \cdot (\epsilon_o \mathbf{E}) = \rho_f + \rho_b$ in macroscopic form, a macroscopic bound charge density[7] $\rho_b = -\nabla \cdot \mathbf{P}$ expressed in terms of $\mathbf{P}$.

- In conclusion, the macroscopic form of Gauss's law can be written for any type of medium as

$$\nabla \cdot \mathbf{D} = \rho_f,$$

with the understanding that $\mathbf{D} \equiv \epsilon_o \mathbf{E} + \mathbf{P}$, "including" the effects of bound charge density $\rho_b = -\nabla \cdot \mathbf{P}$ that may exist within or on the boundaries of the medium.

  - Typically subscript f of ρ_f is dropped in Gauss's law, with the understanding that ρ refers to ρ_f because any non-zero $\rho_b = -\nabla \cdot \mathbf{P}$

[7]Note that if $\mathbf{P}$ is a constant inside a dielectric and zero outside, then $\rho_b = -\nabla \cdot \mathbf{P}$ will be a surface charge density confined to the surface of the dielectric.

effects have already been "lumped" into $\mathbf{D} = \epsilon_o \mathbf{E} + \mathbf{P}$ (as mentioned above).

- The differential form of Gauss's law that we will now write (without the subscript f on ρ_f) as

$$\nabla \cdot \mathbf{D} = \rho, \quad \text{[Gauss's law inside material medium]}$$

appears in integral form (after applying Divergence theorem to volume integral of the differential form) as

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_V \rho dV,$$

where the right side denotes the net *free* charge inside volume V.

- In a large class of dielectric materials macroscopic polarization $\mathbf{P}$ and electric field $\mathbf{E}$ turn out to be linearly related (see Lecture 11) as

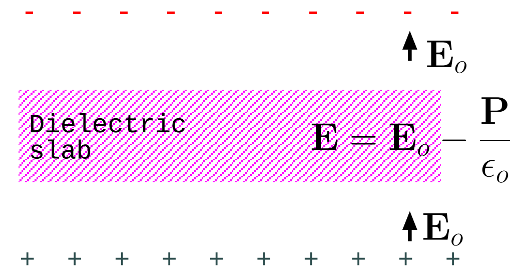
$$\mathbf{P} = \epsilon_o \chi_e \mathbf{E},$$

where $\chi_e \geq 0$ is a dimensionless quantity called **electric susceptibility**. For such materials

$$\mathbf{D} = \epsilon_o \mathbf{E} + \mathbf{P} = \epsilon_o (1 + \chi_e) \mathbf{E} = \epsilon \mathbf{E},$$

where

$$\epsilon = \epsilon_o (1 + \chi_e) \equiv \epsilon_r \epsilon_o$$



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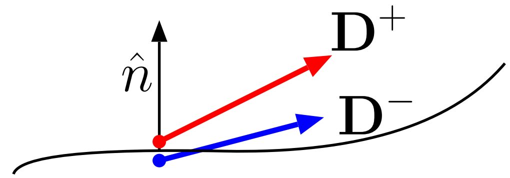
is known as the **permittivity** of the dielectric, and

$$\epsilon_r = 1 + \chi_e$$

its **relative permittivity** or **dielectric constant**.

- Dielectric constant of free space is 1,
  - for air $\epsilon_r \approx 1.0006$,
  - for glass $4 - 10$,
  - dry-to-wet earth $5 - 10$, silicon $11 - 12$, distilled water 81.

  In certain materials χ_e and ϵ are found to be tensors — meaning that $\mathbf{P}$ and $\mathbf{D}$ are no longer aligned with $\mathbf{E}$. Such materials are said to be **anisotropic**, but they will not be studied in this course. Also, there is an exception to the condition $\chi_e \geq 0$ — in collisionless *plasmas* $\chi_e < 0$, as discussed in ECE 350.

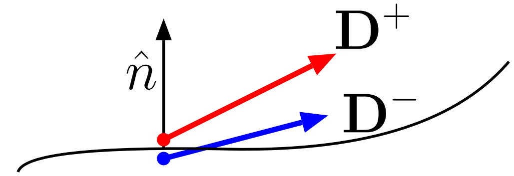


- In Gauss's law $\nabla \cdot \mathbf{D} = \rho$ applicable in material media ρ denotes the free charge carrier density only (after the revisions we have agreed to make).
  - Considering the integral form of Gauss' law applied to a "pillbox" where the right hand side is the total free charge to found inside pillbox, the boundary condition equation relavant for $\mathbf{D} = \epsilon\mathbf{E} + \mathbf{P}$ — see figure in the margin — can be shown to be

$$\hat{n} \cdot (\mathbf{D}^+ - \mathbf{D}^-) = \rho_s$$

(as we have seen before) in general, where ρ_s denotes a surface charge density consisting only of free charge carriers.

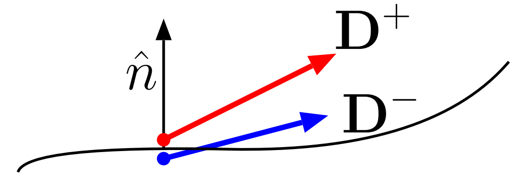
– However, in perfect dielectrics there are no mobile free charge carriers and Gauss's law typically reduces to $\nabla \cdot \mathbf{D} = 0$, while the corresponding **boundary condition** equation for surfaces separating perfect dielectrics becomes



$$\hat{n} \cdot (\mathbf{D}^+ - \mathbf{D}^-) = 0 \quad \Rightarrow \quad D_n^+ = D_n^-,$$

which says that normal component of displacement $\mathbf{D}$ is continuous on such surfaces. This is accompanied by

$$\hat{n} \times (\mathbf{E}^+ - \mathbf{E}^-) = 0 \quad \Rightarrow \quad E_t^+ = E_t^-$$



stating the continuity of tangential components of $\mathbf{E}$, which is universally true as we have seen earlier.