7 Poisson's and Laplace's equations

Summarizing the properties of electrostatic fields we have learned so far, they satisfy the laws of electrostatics shown in the margin and, in addition,

$$\mathbf{E} = -\nabla V \text{ as a consequence of } \nabla \times \mathbf{E} = 0.$$

- Using these relations, we can re-write Gauss's law as

$$\nabla \cdot \mathbf{E} = -\nabla \cdot (\nabla V) = \frac{\rho}{\epsilon_o},$$

from which it follows that

$$\nabla^2 V = -\frac{\rho}{\epsilon_o}, \text{ (Poisson's eqn)}$$

where

$$\nabla^2 V \equiv \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

is known as **Laplacian** of V.

  - A special case of Poisson's equation corresponding to having

$$\rho(x, y, z) = 0$$

everywhere in the region of interest is

$$\nabla^2 V = 0. \text{ (Laplace's eqn)}$$

Laws of electrostatics:

$$\nabla \cdot \mathbf{E} = \rho/\epsilon_o$$
$$\nabla \times \mathbf{E} = 0$$

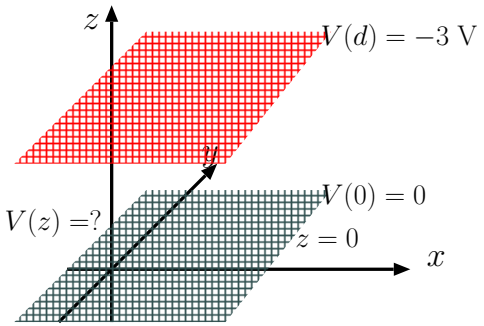
Poisson's eqn:

$$\nabla^2 V = -\frac{\rho}{\epsilon_o}$$

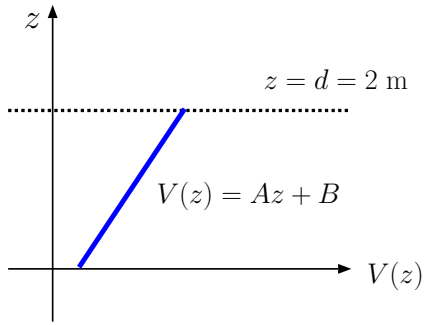
Laplace's eqn:

$$\nabla^2 V = 0$$

Focusing our attention first on Laplace's equation, we note that the equation can be used in charge free-regions to determine the electrostatic potential $V(x, y, z)$ by matching it to specified potentials at boundaries as illustrated in the following examples:



Example 1: Consider a pair of parallel conducting metallic plates of infinite extents in x and y directions but separated in z direction by a finite distance of $d = 2$ m (as shown in the margin). The conducting plates have non-zero surface charge densities (to be determined in Example 2), which are known to be responsible for an electrostatic field $\mathbf{E} = \hat{z}E_z$ measured in between the plates. Each plate has some unique and constant electrostatic potential V since neither $\mathbf{E}(\mathbf{r})$ nor $V(\mathbf{r})$ can dependent the coordinates x or y given the geometry of the problem.



Using Laplace's equation determine $V(z)$ and $\mathbf{E}(z)$ between the plates if the potential of the plate at $z = 0$ is 0 (the ground), while the potential of the plate at $z = d$ is -3 V.

Solution: Since the potential function $V = V(z)$ between the plates is only dependent on z, it follows that Laplace's equation simplifies as

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = \frac{\partial^2 V}{\partial z^2} = 0.$$

This equation can be satisfied by

$$V(z) = Az + B$$

where A and B are constants to be determined. Now applying the given boundary conditions, we first notice that (at the lower plate)

$$V(0) = (Az + B)_{|z=0} = B = 0.$$

Applying the second boundary condition (at the top plate) we find

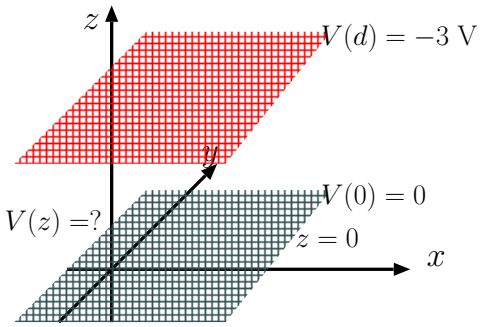
$$V(2) = (Az + 0)_{|z=2} = 2A = -3\,\mathrm{V} \quad \Rightarrow \quad A = -\frac{3}{2}\frac{\mathrm{V}}{\mathrm{m}}.$$

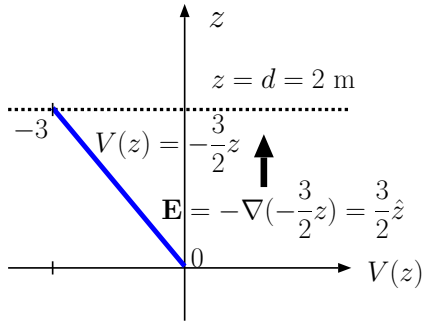
The upshot is, potential function

$$V(z) = -\frac{3}{2}z, \ \text{ for } 0 < z < 2\,\mathrm{m}.$$

Finally, we determine the electric field between the plates as

$$\mathbf{E} = -\nabla V = -\nabla(-\frac{3}{2}z) = \hat{z}\frac{\partial}{\partial z}(\frac{3}{2}z) = \hat{z}\frac{3}{2}\frac{\mathrm{V}}{\mathrm{m}}.$$





Example 2: In Example 1 what are the surface charge densities of the metallic plates located at $z = 0$ and $z = 2$ m surfaces?

Solution: Since the electric field

$$\mathbf{E} = \hat{z}\frac{3}{2}\frac{\mathrm{V}}{\mathrm{m}}$$

in between the plates, comparing this field with the field

$$\mathbf{E} = \hat{z}\frac{\rho_s}{\epsilon_o}$$

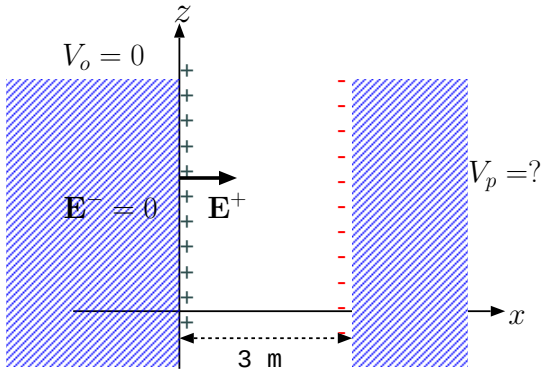
of a pair of parallel surfaces carrying surface charge densities ρ_s and $-\rho_s$ (at $z = 0$ and $z = 2$ m), we find that

$$\rho_s = \frac{3}{2}\epsilon_o$$

on the surface at $z = 0$. The surface at $z = 2$ m has $\rho_s = -\frac{3}{2}\epsilon_o$.

Notice that our solution with equal and opposite charge densities on the parallel surfaces implies that electrostatic fields are zero *within* the conducting plates where the fields due to two charged surfaces are canceling out. This conclusion is consistent with having constant electrostatic potentials within conducting regions as will be discussed in the next lecture.

Example 3: A pair of copper blocks separated by a distance $d = 3$ m in x direction hold surface charge densities of $\rho_s = \pm 2$ C/m^2 on surfaces facing one another as shown in the margin. The blocks are assigned constant potentials $V_o = 0$ and V_p (see figure). What is the potential difference V_p?



Solution: Let $\mathbf{D}^+ = \hat{x}\epsilon_o E_x$ denote the displacement vector in between the blocks, and let $\mathbf{D}^- = 0$ denote the displacement vector *within* the block with a surface at $x = 0$. Then the boundary condition equation used at $x = 0$ implies that

$$\hat{x} \cdot (\mathbf{D}^+ - \mathbf{D}^-) = \epsilon_o E_x = 2\,\frac{\text{C}}{\text{m}^2} \quad \Rightarrow \quad E_x = \frac{2}{\epsilon_o}.$$

In that case, potential difference between the blocks is

$$V = E_x d = \frac{2}{\epsilon_o} 3 = \frac{6}{\epsilon_o}.$$

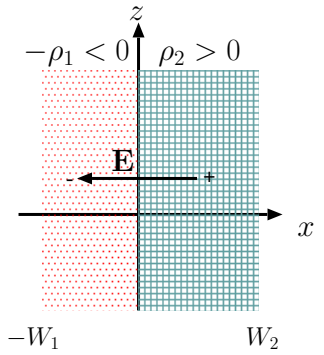
Since the block on the left is at a higher potential (electric field vectors point from high to low potential) assigned as $V_o = 0$, we must have

$$V_p = -\frac{6}{\epsilon_o}.$$

Poisson's equation

$$\nabla^2 V = -\frac{\rho}{\epsilon_o}$$

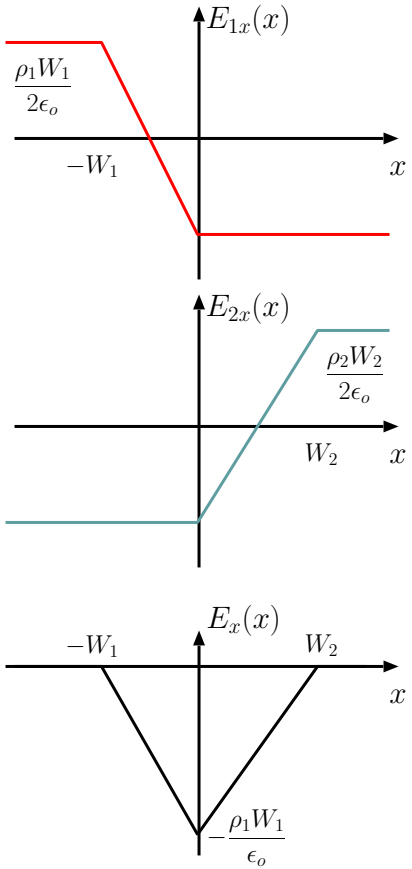
is used in regions where the charge density $\rho(\mathbf{r})$ is non-zero. The following example illustrates a possible use of Poisson's equation.



Example 4: An infinite charged slab of width W_1, located over $-W_1 < x < 0$, has a negative volumetric charge density of $-\rho_1$ C/m^3, $\rho_1 > 0$. A second slab of width W_2 and positive charge density ρ_2 is located over $0 < x < W_2$ as shown in the margin. The electric field of this static charge configuration under the constraint $W_1\rho_1 = W_2\rho_2$ was computed in an earlier section as

$$\mathbf{E} = \begin{cases} -\hat{x}\frac{\rho_1(x+W_1)}{\epsilon_o}, & \text{for } -W_1 < x < 0 \\ \hat{x}\frac{\rho_2(x-W_2)}{\epsilon_o}, & \text{for } 0 < x < W_2 \end{cases}$$

and is depicted in the margin. Determine the electrostatic potential in the region and the potential difference $V_{21} \equiv V(W_2) - V(-W_1)$ satisfying Poisson's equation.



Solution: This is a one dimensional geometry where $\mathbf{E}$ and potential V depend only on coordinate x. Therefore, Poisson's equation $\nabla^2 V = -\rho/\epsilon_o$ takes the simplified form

$$\frac{d^2V}{dx^2} = -\frac{\rho(x)}{\epsilon_o}.$$

Integral of this equation over x yields in the left $\frac{dV}{dx} = -E_x$, which implies, given the electric field result from above,

$$\frac{dV}{dx} = \begin{cases} \frac{\rho_1(x+W_1)}{\epsilon_o}, & \text{for } -W_1 < x < 0 \\ -\frac{\rho_2(x-W_2)}{\epsilon_o}, & \text{for } 0 < x < W_2 \end{cases}$$

Integrating $\frac{dV}{dx}$ once more (i.e., finding suitable anti-derivatives with integration constants), we find

$$V(x) = \begin{cases} \frac{\rho_1(x+W_1)^2}{2\epsilon_o} + V_1, & \text{for } -W_1 < x < 0 \\ -\frac{\rho_2(x-W_2)^2}{2\epsilon_o} + V_2, & \text{for } 0 < x < W_2 \end{cases}$$

where the integration constants included *on each line* have been selected so that $V_2 = V(W_2)$, $V_1 = V(-W_1)$.

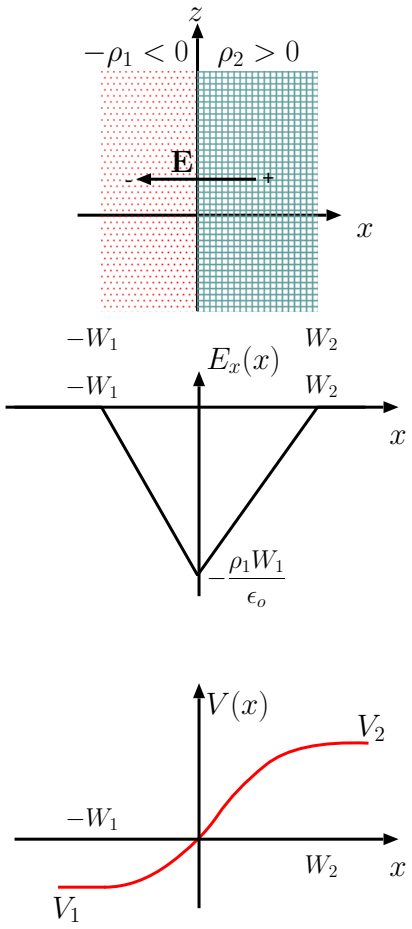
Requiring a unique potential value at $x = 0$ (we can only associate a single potential energy level with each position in space) compatible with this expression for $V(x)$, we obtain

$$\frac{\rho_1(0+W_1)^2}{2\epsilon_o} + V_1 = -\frac{\rho_2(0-W_2)^2}{2\epsilon_o} + V_2,$$

from which

$$V_{21} = V_2 - V_1 = \frac{\rho_2 W_2^2 + \rho_1 W_1^2}{2\epsilon_o} = \frac{\rho_2 W_2(W_1+W_2)}{2\epsilon_o} = \frac{\rho_1 W_1(W_1+W_2)}{2\epsilon_o}.$$

Note that the equation above can be solved for W_1, W_2, and $W_2 + W_1$ in terms of V_{12}, ρ_2, and ρ_1, providing useful formulas for diode design (see ECE 440). We can also get useful specific formulae for V_1 and V_2 by imposing $V(0) = 0$, i.e., choosing $x = 0$ to be the reference point.



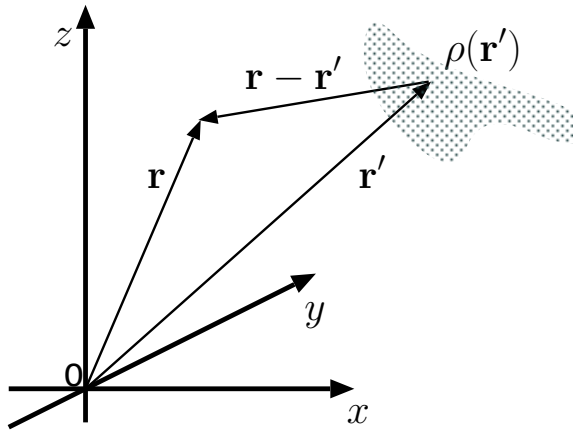
- The solution of Poisson's equation

$$\nabla^2 V = -\frac{\rho}{\epsilon_o}$$

with an arbitrary ρ existing over a finite region in space can be obtained as

$$V(\mathbf{r}) = \int \frac{\rho(\mathbf{r}')}{4\pi\epsilon_o|\mathbf{r}-\mathbf{r}'|} d^3\mathbf{r}'$$

where $d^3\mathbf{r}' \equiv dx'dy'dz'$ and the 3D integral on the right over the primed coordinates is performed over the entire region where the charge density is non-zero.



- **Verification:** The solution above can be verified by combining a number of results we have seen earlier on:

  1. In Lecture 5 we learned that the electric potential $V(\mathbf{r})$ of a point charge e at the origin is

  $$V(\mathbf{r}) = \frac{e}{4\pi\epsilon_o|\mathbf{r}|}.$$

  Clearly, this singular result is a solution of Poisson's equation above (and the stated boundary condition) for a charge density input of

  $$\rho(\mathbf{r}) = e\delta(\mathbf{r}).$$

  2. Using ECE 210-like terminology and notation, the above result can be represented as

  $$\delta(\mathbf{r}) \rightarrow \boxed{\text{Poisson's Eqn}} \rightarrow \frac{1}{4\pi\epsilon_o|\mathbf{r}|}$$

identifying the output on the right as 3D "impulse response" of the **linear** and **shift-invariant** (LSI) system represented by Poisson's equation.

3. Because of shift-invariance, we have

$$\delta(\mathbf{r}-\mathbf{r}') \rightarrow \boxed{\text{Poisson's Eqn}} \rightarrow \frac{1}{4\pi\epsilon_o|\mathbf{r}-\mathbf{r}'|},$$

meaning that a shifted impulse causes a shifted impulse response.

The shifted impulse response is usually called "Green's function" $G(\mathbf{r},\mathbf{r}')$ in EM theory.

4. Because of linearity, we are allowed to use superpositioning arguments like

$$\int \rho(\mathbf{r}')\delta(\mathbf{r}-\mathbf{r}')d^3\mathbf{r}' = \rho(\mathbf{r}) \rightarrow \boxed{\text{Poisson's Eqn}} \rightarrow \int \rho(\mathbf{r}')\frac{1}{4\pi\epsilon_o|\mathbf{r}-\mathbf{r}'|}d^3\mathbf{r}' = V(\mathbf{r}),$$

which concludes our verification of the electrostatic[1] potential solution. Note how we made use of the *sifting property* of the impulse (from ECE 210) in above calculation.

[1]Also, in quasi-statics we use $\rho(\mathbf{r}',t)$ to obtain $V(\mathbf{r},t)$ over regions small compared to $\lambda = c/f$, with f the highest frequency in $\rho(\mathbf{r}',t)$.

- As an application of the general solution of Poisson's equation, namely

$$\nabla^2 V = -\frac{\rho}{\epsilon_o} \;\; \Rightarrow \;\; V(\mathbf{r}) = \int \frac{\rho(\mathbf{r}')}{4\pi\epsilon_o|\mathbf{r}-\mathbf{r}'|} d^3\mathbf{r}',$$

we next provide an *outline* of the proof of **Helmholtz theorem** (see Lecture 4) which states that *any vector field* $\mathbf{F}(x,y,z)$ *that vanishes in the limit* $r = \sqrt{x^2+y^2+z^2} \to \infty$ *can be reconstructed uniquely from its* divergence *and* curl*:*

  - First, with no loss of generality, we write

$$\mathbf{F} = -\nabla V + \nabla \times \mathbf{A}$$

    in terms of scalar and vector fields $V(x,y,z)$ and $\mathbf{A}(x,y,z)$ *to be identified* as follows[2]:

  - Taking first the divergence of $\mathbf{F}$ (and using $\nabla \cdot \nabla \times \mathbf{A} = 0$), we find that

$$\nabla \cdot \mathbf{F} = -\nabla^2 V \;\; \Rightarrow \;\; V(\mathbf{r}) = \int \frac{\nabla' \cdot \mathbf{F}(\mathbf{r}')}{4\pi|\mathbf{r}-\mathbf{r}'|} d^3\mathbf{r}'$$

    in analogy with Poisson's equation (with $\nabla' \cdot \mathbf{F}(\mathbf{r}')$ replacing $\rho(\mathbf{r}')/\epsilon_o$ where ∇' is "del" in (x',y',z')-space).

[2]This is possible because of the vector identity $-\nabla^2\mathbf{G} = \nabla \times (\nabla \times \mathbf{G}) - \nabla(\nabla \cdot \mathbf{G})$ — call $-\nabla^2\mathbf{G} \equiv \mathbf{F}$, which, according to this identity, is equal to the curl of a *vector* $\nabla \times \mathbf{G} \equiv \mathbf{A}$ (with $\nabla \cdot \mathbf{A} = \nabla \cdot \nabla \times \mathbf{G} = 0$), minus the gradient of a *scalar* $\nabla \cdot \mathbf{G} \equiv V$, as claimed. The challenge is in figuring out the underlying $\mathbf{G}$ for a given $\mathbf{F}$, which is what *Helmholtz theorem* is all about.

– Likewise, the curl of $\mathbf{F}$ (with $\nabla \times \nabla V = 0$) leads us to, with a divergence-free [3] $\mathbf{A}$, to

$$\nabla\times\mathbf{F} = \nabla\times\nabla\times\mathbf{A} = \nabla(\nabla\cdot\mathbf{A})-\nabla^2\mathbf{A} = -\nabla^2\mathbf{A} \;\; \Rightarrow \;\; \mathbf{A}(\mathbf{r}) = \int \frac{\nabla' \times \mathbf{F}(\mathbf{r}')}{4\pi|\mathbf{r}-\mathbf{r}'|} d^3\mathbf{r}'$$

once again in analogy with Poisson's equation[4].

These results validate Helmholtz theorem for fields $\mathbf{F}$ vanishing at infinity, since, *evidently,* V and $\mathbf{A}$ needed to reconstruct $\mathbf{F}$ can be uniquely specified in terms of $\nabla \cdot \mathbf{F}$ and $\nabla \times \mathbf{F}$, respectively.

[3] To confirm $\nabla \cdot \mathbf{A} = 0$ directly, use the identity $\nabla \cdot (\alpha\mathbf{G}) = \alpha\nabla \cdot \mathbf{G} + \mathbf{G} \cdot \nabla\alpha$ to expand $\mathbf{A}(\mathbf{r})$ as

$$\nabla\cdot\int \frac{\nabla' \times \mathbf{F}(\mathbf{r}')}{4\pi|\mathbf{r}-\mathbf{r}'|} d^3\mathbf{r}' = \int \frac{\nabla' \times \mathbf{F}(\mathbf{r}')}{4\pi}\cdot\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|} d^3\mathbf{r}' = -\int \frac{\nabla' \times \mathbf{F}(\mathbf{r}')}{4\pi}\cdot\nabla'\frac{1}{|\mathbf{r}-\mathbf{r}'|} d^3\mathbf{r}' = \int \frac{\nabla' \cdot (\nabla' \times \mathbf{F}(\mathbf{r}'))}{4\pi|\mathbf{r}-\mathbf{r}'|} d^3\mathbf{r}' = 0,$$

after also using integration by parts for an integrand that vanishes as $|\mathbf{r}| \to \infty$ and a symmetry relation $\nabla|\mathbf{r}-\mathbf{r}'|^{-1} = -\nabla'|\mathbf{r}-\mathbf{r}'|^{-1}$ which is easy to confirm.

[4] While the vector field $\mathbf{A}$ identified above is divergence-free, $\nabla\times\mathbf{A}$ in the $\mathbf{F} = -\nabla V + \nabla\times\mathbf{A}$ expansion can also be replaced with $\nabla\times\mathbf{A}'$ so long as $\mathbf{A}' = \mathbf{A}+\nabla\Psi$ since $\nabla\times\nabla\Psi$ is unconditionally zero independent of the choice of Ψ. Note it is possible to specify Ψ so that $\nabla \cdot \mathbf{A}' = \nabla \cdot \nabla\Psi = \nabla^2\Psi \neq 0$ in which case $\mathbf{A}'$ will be a divergent solution of the $\nabla \times \mathbf{F}$ equation above! The additive term $\nabla\Psi$ in $\mathbf{A}'$ is analogous to allowing a constant number to be added to V! The freedom to specify Ψ and thus $\nabla \cdot \mathbf{A}'$ at will is known as *gauge freedom* and any choice of Ψ making $\nabla \cdot \mathbf{A}' = 0$ is known as *Coulomb's gauge.*