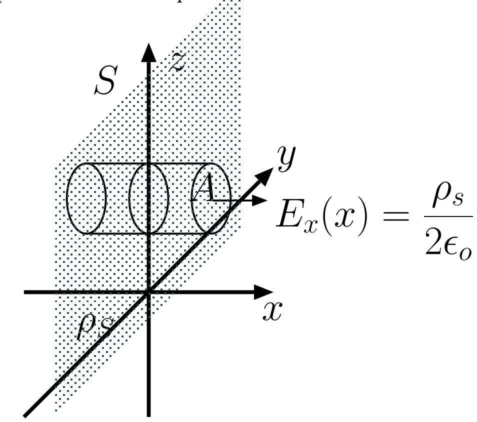
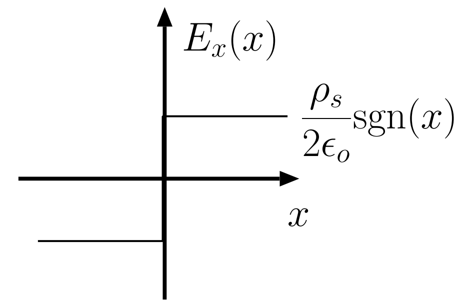
3 Gauss's law and static charge densities

We continue with examples illustrating the use of Gauss's law in macroscopic field calculations:





Example 1: Point charges Q are distributed over $x = 0$ plane with an average surface charge density of ρ_s C/m^2. Determine the macroscopic electric field $\mathbf{E}$ of this charge distribution using Gauss's law.

Solution: First, invoking Coulomb's law, we convince ourselves that the field produced by surface charge density ρ_s C/m^2 on $x = 0$ plane will be of the form $\mathbf{E} = \hat{x}E_x(x)$ where $E_x(x)$ is an odd function of x because y- and z-components of the field will cancel out due to the symmetry of the charge distribution. In that case we can apply Gauss's law over a cylindrical integration surface S having circular caps of area A parallel to $x = 0$, and obtain

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q_V \quad \Rightarrow \quad \epsilon_o E_x(x)A - \epsilon_o E_x(-x)A = A\rho_s,$$

which leads, with $E_x(-x) = -E_x(x)$, to

$$E_x(x) = \frac{\rho_s}{2\epsilon_o} \text{ for } x > 0.$$

Hence, in vector form

$$\mathbf{E} = \hat{x}\frac{\rho_s}{2\epsilon_o}\text{sgn}(x),$$

where sgn(x) is the signum function, equal to ± 1 for $x \gtrless 0$.

Note that the macroscopic field calculated above is discontinuous at $x = 0$ plane containing the surface charge ρ_s, and points away from the same surface on both sides.

Example 2: Point charges Q are distributed throughout an infinite slab of width W located over $-\frac{W}{2} < x < \frac{W}{2}$ with an average charge density of ρ C/m^3. Determine the macroscopic electric field $\mathbf{E}$ of the charged slab inside and outside.

Solution: Symmetry arguments based on Coulomb's law once again indicates that we expect a solution of the form $\mathbf{E} = \hat{x}E_x(x)$ where $E_x(x)$ is an odd function of x.

In that case, applying Gauss's law with a cylindrical surface S having circular caps of area A parallel to $x = 0$ extending between $-x$ and $x < \frac{W}{2}$, we obtain

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q_V \Rightarrow \epsilon_o E_x(x)A - \epsilon_o E_x(-x)A = \rho 2xA,$$

which leads, with $E_x(-x) = -E_x(x)$, to

$$E_x(x) = \frac{\rho x}{\epsilon_o} \text{ for } 0 < x < \frac{W}{2}.$$

For $x > \frac{W}{2}$,

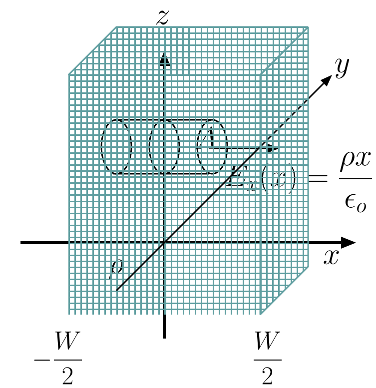
$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q_V \Rightarrow \epsilon_o E_x(x)A - \epsilon_o E_x(-x)A = AW\rho,$$

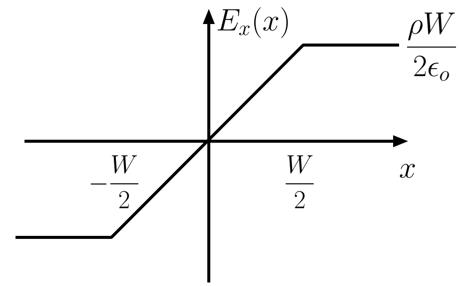
leading to

$$E_x(x) = \frac{\rho W}{2\epsilon_o} \text{ for } x > \frac{W}{2}.$$

These results can be combined as

$$\mathbf{E} = \hat{x}E_x(x) = \begin{cases} -\hat{x}\frac{\rho W}{2\epsilon_o}, & \text{for } x < -\frac{W}{2} \\ \hat{x}\frac{\rho x}{\epsilon_o}, & \text{for } -\frac{W}{2} < x < \frac{W}{2} \\ \hat{x}\frac{\rho W}{2\epsilon_o}, & \text{for } x > \frac{W}{2}. \end{cases}$$





Note that the field solution depicted in the margin in terms of $E_x(x)$ plot is a continuous function of x as opposed to the discontinuous $E_x(x)$ solution obtained in Example 1 for the macroscopic field of a surface charge.

- In future calculations of electrostatic fields, we can use our previous results, namely

  - Coulomb field
  $$\mathbf{E} = \hat{r}\frac{Q}{4\pi\epsilon_o r^2} \text{ of a point charge } Q,$$
  - Field
  $$\mathbf{E} = \hat{r}\frac{\lambda}{2\pi\epsilon_o r} \text{ of constant line density } \lambda,$$
  - Field
  $$\mathbf{E} = \hat{x}\frac{\rho_s}{2\epsilon_o}\text{sgn}(x) \text{ of constant surface density } \rho_s,$$
  - Field
  $$\mathbf{E} = \hat{x}\frac{\rho x}{\epsilon_o} \text{ of constant volume density } \rho$$

  as building blocks — that is, the above field equations can be superposed to determine the field structure of charge distributions $\rho(x, y, z)$ that can be expressed as superpositions of simpler charge distributions with known field structures. Some examples...

Example 3: Consider a pair of surface charges $\rho_s > 0$ and $-\rho_s$ $\mathrm{C/m^2}$ of equal magnitudes placed on $x = -\frac{W}{2}$ and $x = \frac{W}{2}$ surfaces. Determine the electric field of this charge distribution depicted in the margin.

Solution: The field of charge density ρ_s $\mathrm{C/m^2}$ on $x = -\frac{W}{2}$ plane should be

$$\mathbf{E}_+ = \hat{x}\frac{\rho_s}{2\epsilon_o}\mathrm{sgn}(x + \frac{W}{2}),$$

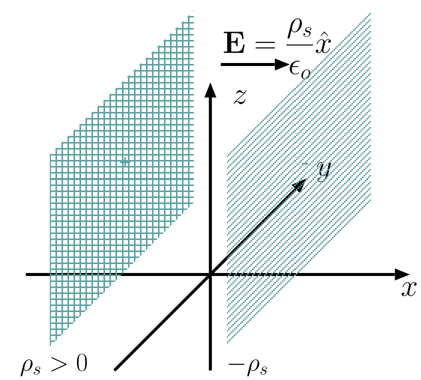
pointing away from the discontinuity surface at $x = -\frac{W}{2}$ on both sides. Likewise, the field of charge density $-\rho_s$ $\mathrm{C/m^2}$ on $x = \frac{W}{2}$ plane should be

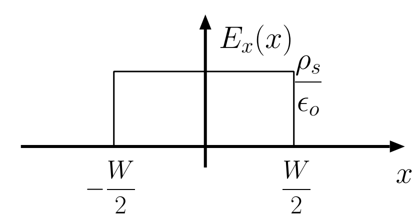
$$\mathbf{E}_- = -\hat{x}\frac{\rho_s}{2\epsilon_o}\mathrm{sgn}(x - \frac{W}{2}),$$

pointing toward $x = \frac{W}{2}$ surface from both sides. Superposing the two fields, we find that

$$\mathbf{E} = \mathbf{E}_+ + \mathbf{E}_- = \begin{cases} \hat{x}\frac{\rho_s}{\epsilon_o}, & \text{for } -\frac{W}{2} < x < \frac{W}{2}, \\ 0, & \text{otherwise,} \end{cases} = \hat{x}\frac{\rho_s}{\epsilon_o}\mathrm{rect}(\frac{x}{W})$$

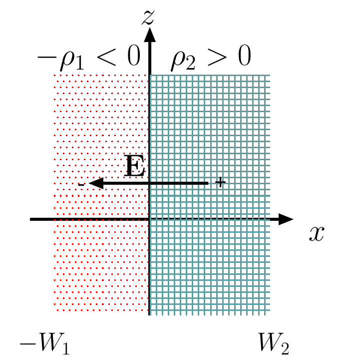
as depicted in the margin.





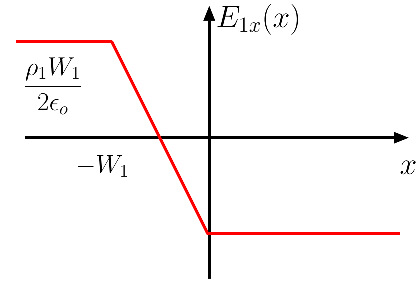
Note that the field lines of our solution point from positive charges on one surface to the negative charges resting on the other surface — this field has the structure of fields encountered in parallel plate capacitors that we will be studying soon.

Example 4: An infinite charged slab of width W_1, located over $-W_1 < x < 0$, has a negative volumetric charge density of $-\rho_1$ C/m^3, $\rho_1 > 0$. A second slab of width W_2 and positive charge density ρ_2 is located over $0 < x < W_2$ as shown in the margin. Compute the electric field of this static charge configuration if $W_1\rho_1 = W_2\rho_2$, implying that the entire system is charge neutral (i.e., a net charge of zero).



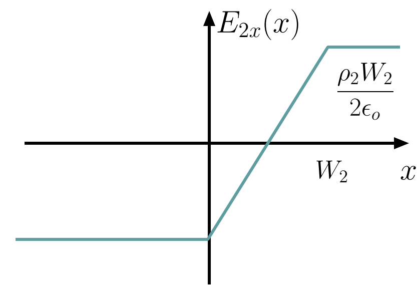
Solution: We note that the field of slab W_1 can be written as

$$\mathbf{E}_1 = \begin{cases} \hat{x}\frac{\rho_1 W_1}{2\epsilon_o}, & \text{for } x < -W_1 \\ -\hat{x}\frac{\rho_1(x+\frac{W_1}{2})}{\epsilon_o}, & \text{for } -W_1 < x < 0 \\ -\hat{x}\frac{\rho_1 W_1}{2\epsilon_o}, & \text{for } x > 0 \end{cases}$$



as depicted in the margin. Likewise, the field of slab W_2 is

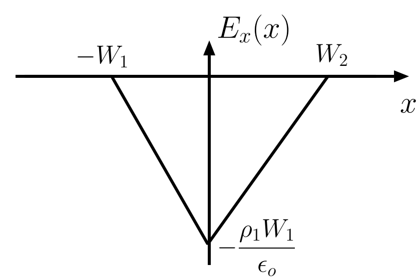
$$\mathbf{E}_2 = \begin{cases} -\hat{x}\frac{\rho_2 W_2}{2\epsilon_o}, & \text{for } x < 0 \\ \hat{x}\frac{\rho_2(x-\frac{W_2}{2})}{\epsilon_o}, & \text{for } 0 < x < W_2 \\ \hat{x}\frac{\rho_2 W_2}{2\epsilon_o}, & \text{for } x > W_2. \end{cases}$$



Note that field strengths $\frac{\rho_1 W_1}{2\epsilon_o}$ and $\frac{\rho_2 W_2}{2\epsilon_o}$ showing up in the expressions for $\mathbf{E}_1$ and $\mathbf{E}_2$ are equal because of the charge neutrality condition $W_1\rho_1 = W_2\rho_2$.

Consequently, when we superpose $\mathbf{E}_1$ and $\mathbf{E}_2$, the fields cancel out outside the region $-W_1 < x < W_2$, so that the total field becomes (as depicted in the margin)

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 = \begin{cases} -\hat{x}\frac{\rho_1(x+W_1)}{\epsilon_o}, & \text{for } -W_1 < x < 0 \\ \hat{x}\frac{\rho_2(x-W_2)}{\epsilon_o}, & \text{for } 0 < x < W_2 \\ 0, & \text{otherwise.} \end{cases}$$



- **Charge density** formalism which we find convenient to use for macroscopic field calculations can also be "adjusted" to describe the distributions of **isolated point charges** via the use of impulses or *delta functions* in space.

  - For example

    $$\rho(x,y,z) = Q\delta(x-x_o)\delta(y-y_o)\delta(z-z_o)$$

    can be regarded as a 3D volumetric charge density function representing a point charge Q located at a coordinate

    $$\mathbf{r} = (x,y,z) = (x_o,y_o,z_o) \equiv \mathbf{r}_o.$$

    - This is justified because we can regard $\delta(x-x_o)$ to be zero everywhere except at $x = x_o$. By extension, the product

      $$\delta(x-x_o)\delta(y-y_o)\delta(z-z_o)$$

      is zero everywhere except at $\mathbf{r} = \mathbf{r}_o = (x_o,y_o,z_o)$ — therefore the density function $\rho(x,y,z)$ defined above behaves correctly to indicate the absence of charges everywhere with the exception of $\mathbf{r}_o$. Furthermore, the area property of the impulse implies that the volume integral of the charge density yields

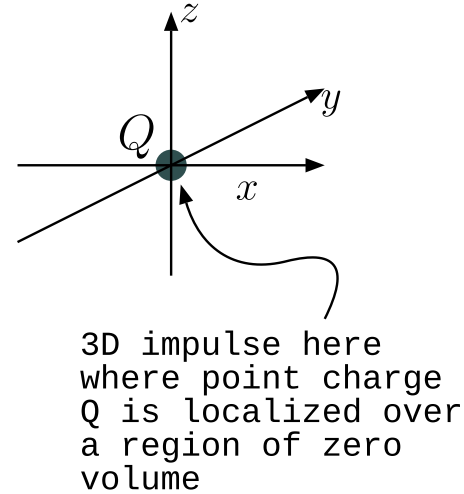
      $$\int \rho dV = \int\int\int Q\delta(x-x_o)\delta(y-y_o)\delta(z-z_o)dxdydz = Q$$

      as it should.

Gauss' Law in terms of charge density:

$$\oint_S \mathbf{D}\cdot d\mathbf{S} = \int_V \rho dV$$

$$\rho(x,y,z) = Q\delta(x)\delta(y)\delta(z)$$



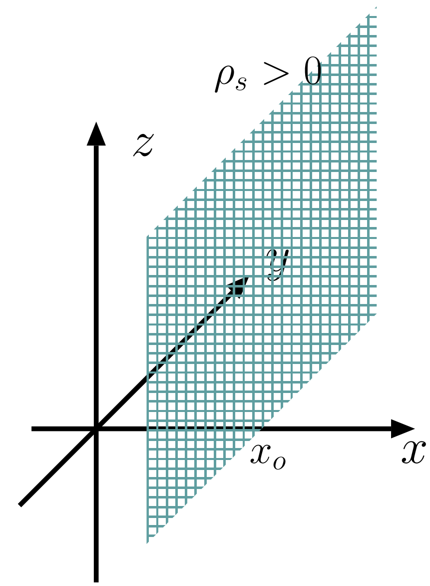
- $\circ$ Notice that the shifted impulses $\delta(x - x_o)$, etc., must have m^{-1} units in order to maintain dimensional consistency in the above expression.

– Another example is

$$\rho(x, y, z) = \rho_s(y, z)\delta(x - x_o)$$

representing a surface charge density of $\rho_s(y, z)$ C/m^2 on $x = x_o$ plane.

$\rho(x, y, z) = \rho_s(y, z)\delta(x - x_o)$

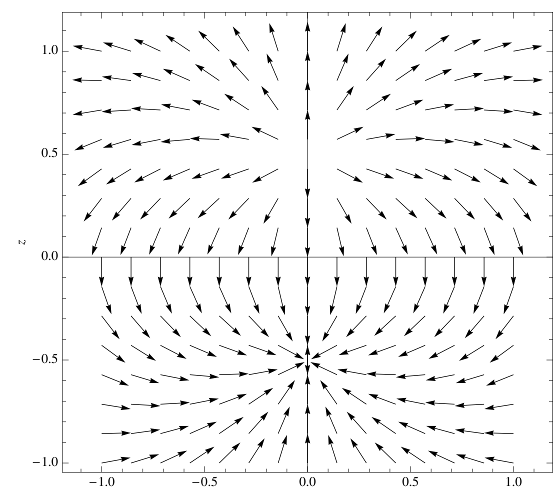


Example 5: Figure in the margin depicts (for the $d = 1$) the $\hat{E}$-field of a pair of charges $\pm Q$ located at $(0, 0, \pm\frac{d}{2})$ derived from

$$\mathbf{E}(\mathbf{r}) = \frac{Q(\mathbf{r} - \frac{d}{2}\hat{z})}{4\pi\epsilon_o|\mathbf{r} - \frac{d}{2}\hat{z}|^3} + \frac{-Q(\mathbf{r} + \frac{d}{2}\hat{z})}{4\pi\epsilon_o|\mathbf{r} + \frac{d}{2}\hat{z}|^3}$$

$$= \frac{Q}{4\pi\epsilon_o}[\frac{(x, y, z - \frac{d}{2})}{|(x, y, z - \frac{d}{2})|^3} - \frac{(x, y, z + \frac{d}{2})}{|(x, y, z + \frac{d}{2})|^3}]\ \mathrm{V/m}.$$

Determine the electric flux $\int_{xy} \mathbf{E} \cdot d\mathbf{S}$ across the entire xy-plane using $d\mathbf{S} = -\hat{z}dxdy$.



Solution: Because of linearity, the flux we want to calculate equals the sum of the flux due to charge Q at $(0, 0, \frac{d}{2})$ above xy-plane and the flux due to charge $-Q$ at $(0, 0, -\frac{d}{2})$ above xy-plane.

Since by Gauss's law $\oint_S \mathbf{E}\cdot d\mathbf{S} = \frac{Q}{\epsilon_o}$ for any S surrounding Q, we can, by symmetry, infer that

$$\int_{xy} \mathbf{E}\cdot(-\hat{z}dxdy) = \frac{Q}{2\epsilon_o}$$

when only charge Q is considered — the logic here is, half of flux $\oint_S \mathbf{E}\cdot d\mathbf{S} = \frac{Q}{\epsilon_o}$ emanating from charge Q should go up and the remaining half should go down crossing the xy-plane in downward direction. Likewise, since $\oint_S \mathbf{E}\cdot d\mathbf{S} = \frac{-Q}{\epsilon_o}$ for any S surrounding $-Q$, again by symmetry, we can infer

$$\int_{xy} \mathbf{E}\cdot(-\hat{z}dxdy) = \frac{Q}{2\epsilon_o}$$

due to charge $-Q$ only — the logic in this case is, half of flux $\frac{Q}{\epsilon_o}$ "entering" charge $-Q$ is "coming from" above crossing the xy-plane in downward direction.

Thus, by superposition, we find total

$$\int_{xy} \mathbf{E}\cdot(-\hat{z}dxdy) = \frac{Q}{2\epsilon_o} + \frac{Q}{2\epsilon_o} = \frac{Q}{\epsilon_o}.$$

The above result can be *confirmed directly* by evaluating the integral

$$\begin{aligned}\int_{xy} \mathbf{E}(x,y,0)\cdot(-\hat{z}dxdy) &= \int_{xy} \frac{Q}{4\pi\epsilon_o}\left[\frac{(x,y,-\frac{d}{2})}{|(x,y,-\frac{d}{2})|^3} - \frac{(x,y,\frac{d}{2})}{|(x,y,\frac{d}{2})|^3}\right]\cdot(-\hat{z}dxdy) \\ &= \frac{Q}{4\pi\epsilon_o}\int_{xy} \frac{d}{|(x,y,-\frac{d}{2})|^3}dxdy = \frac{Qd}{2\epsilon_o}\int_{r=0}^{\infty} \frac{r}{(r^2+(\frac{d}{2})^2)^{3/2}}dr \\ &= \frac{Q}{\epsilon_o}.\end{aligned}$$

Just before the last step we have replaced $dxdy$ by $rdrd\phi$, where $r \equiv \sqrt{x^2+y^2}$, and carried out the ϕ integration before completing the r integration as a last step (which you should verify).