Fields and waves in nature and engineering — the big picture:

Fundamental building blocks of matter — electrons and protons at atomic scales — interact with one another gravitationally and via "electromagnetic" forces. These interactions are most conveniently described in terms of suitably defined "vector fields" that permeate space and time, or simply the *space-time* $(x, y, z, t) \equiv (\mathbf{r}, t)$. Interactions attributed to particle *masses* can be formulated by *gravitational fields* $\mathbf{g}(\mathbf{r}, t)$ specified in reference frames where spatial coordinates $\mathbf{r} = (x, y, z)$ are defined. Far stronger interactions attributed to particle *charges*, on the other hand, are formulated in terms of a *pair* of vector fields, $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$, known as *electric and magnetic fields*, respectively.

Electric and magnetic fields:

A particle with charge q and mass m as well as position and velocity vectors $\mathbf{r}$ and $\mathbf{v} = \frac{d\mathbf{r}}{dt}$ specified at an instant t *within* a measurement frame (or "lab" frame) will be accelerated in accordance with

$$m\frac{d\mathbf{v}}{dt} = q(\mathbf{E}(\mathbf{r}, t) + \mathbf{v} \times \mathbf{B}(\mathbf{r}, t)), \tag{1}$$

which is *Newton's 2nd law of motion*[1] for a particle under the influence of *Lorentz force*

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}). \tag{2}$$

In view of (1), the operational definitions of fields $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$ arise from particle acceleration $\mathbf{a} = \frac{d\mathbf{v}}{dt}$ that can be measured in the lab frame: the electric field $\mathbf{E}$ is evidently force per unit stationary charge (i.e., $\mathbf{v} = 0$) whereas field $\mathbf{B}$ describes an additional force per charge in transport (i.e., $q\mathbf{v}$) that acts in a direction perpendicular to $\mathbf{v}$.

There are important differences between gravitational and electromagnetic interactions: Gravitational interactions are always attractive indicating that particle masses m that generate the gravitational field $\mathbf{g}(\mathbf{r}, t)$ must all have the same algebraic sign (taken to be positive by convention). Electromagnetic interactions, on the other hand, are attractive or repulsive depending on particle charges q which can be positive or negative. *By convention* a positive charge $q = e \approx 1.6 \times 10^{-19}$ C is attributed to the fundamental particle know as *proton*, while, again by convention, $q = -e$ for an *electron*, the sole companion of the proton within a hydrogen atom[2]. Protons and electrons are charged elementary building blocks[3] of all atoms (hydrogen as

[1]Valid so long as $|\mathbf{v}| \ll c$ where c is the speed of light in vacuum.

[2]Hydrogen atom exists as a consequence of mutual attraction between proton and electron counterbalanced by quantum mechanical constraints on allowed energy states — the constraints include the influence of short-lived *virtual* particle/anti-particle pairs interacting with the proton and electron in a sporadic manner.

[3]Atoms can also contain in their nuclei varying numbers of an uncharged particle known as the *neutron* which is responsible for different isotopes of chemical elements (e.g., the hydrogen isotope known as deuterium contains a neutron in addition to a proton and an electron). While neutrons have no net

well as atoms of heavier elements) that constitute the matter around us. In a collection of fundamental particles the total mass is always a monotonically increasing function of the number of particles. However, that is not the case with total charge since individual particle charges can be positive or negative. In fact, the net charge density $\rho(\mathbf{r}, t)$ found in macroscopic amounts of matter is typically close to zero as a result of having nearly equal numbers of protons and electrons in ordinary matter composed of charge-neutral atoms and molecules[4].

charge, they consist of charged sub-nuclear particles known as up ($\frac{2}{3}e$) and down ($-\frac{1}{3}e$) *quarks* whose motions within the neutron establish currents and a magnetic moment.

[4]The reason why intrinsically weaker gravity becomes dominant in the macro world.

Fields are relative:

Physical laws that we use today to describe our surroundings have been developed to have identical forms in all reference frames in uniform motion with respect to one another. For instance, Lorentz force law on a charge q is

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \text{ and } \mathbf{F}' = q(\mathbf{E}' + \mathbf{v}' \times \mathbf{B}') \tag{3}$$

in terms of unprimed and primed variables measured in two reference frames. Moreover, particle charge q and the speed of light c are assigned invariant[5] values in reference frames in relative motion (thus q' and c' are unnecessary to invoke in physical models). The ramifications of these restrictions constituting the **special theory of relativity** (first described by Einstein in 1905 and covered at UIUC in PHYS 325) are in full accord with experimental measurements. They are also well matched by **Newtonian relations** (approximate but more intuitive laws of dynamics covered in PHYS 211) if and when the relative speed of primed and unprimed frames is negligible compared to the speed of light c.

Since in Newtonian descriptions mass m and acceleration $\frac{d\mathbf{v}}{dt}$ have invariant values in all reference frames, it follows that if and when $|\mathbf{v}' - \mathbf{v}| \ll c$, then $\mathbf{F}' = \mathbf{F}$, in which case (3) implies

$$\mathbf{E}' + \mathbf{v}' \times \mathbf{B}' = \mathbf{E} + \mathbf{v} \times \mathbf{B}. \tag{4}$$

[5]Other "relativistic invariants" between different reference frames include particle (rest) masses and the so-called "spacetime interval" $\sqrt{t^2 - L^2/c^2} = \sqrt{t'^2 - L'^2/c^2}$ between two events occurring at two locations and two times separated by a distance L and time-delay t, respectively. Relativistic invariants are the most prized physical quantitites to focus on in relativistic models (simply because they remain fixed in all reference frames). Note that distances $L \neq L'$ and time-delays $t \neq t'$ are not *relativistic* invariants!

Then, for a stationary charge in the primed frame, we have $\mathbf{v}' = 0$ and

$$\mathbf{E}' = \mathbf{E} + \mathbf{v} \times \mathbf{B}, \tag{5}$$

which indicates that force per unit stationary charge in the primed frame — i.e., the electric field in the primed frame — is a linear combination of electrical and magnetic forces exerted on the same charge as seen from another reference frame (unprimed) where the charge appears to have a vector velocity $\mathbf{v}$.

Thus, electric and magnetic fields needed in the formulation of charged particle interactions are not unrelated to one another — rather, they intermix in a manner that depends on the reference frame[6] being used for analysis purposes. Note that charges q which are stationary in one reference frame (and therefore carry no electrical current) will appear to be in motion in another frame and thus carry electrical currents I. It must therefore be evident that the equations for $\mathbf{E}$ and $\mathbf{B}$ in any reference frame must be cross-coupled and depend on both charge and current densities that are measured in the same frame.

[6]Given $\mathbf{E}$ and $\mathbf{B}$ measured in the lab, $\mathbf{E}'$ and $\mathbf{B}'$ measured by an observer moving through the lab with a constant velocity $\mathbf{v}$ are well approximated by $\mathbf{E}' \approx \mathbf{E} + \mathbf{v} \times \mathbf{B}$ and $\mathbf{B}' \approx \mathbf{B} - \frac{\mathbf{v}\times\mathbf{E}}{c^2}$ so long as $|\mathbf{v}| \ll c = 3 \times 10^8$ m/s, the *speed of light* in free space (shown by relativistic analysis discussed in PHYS 225 — exact transformation formulae are $\mathbf{E}'_{\|} = \mathbf{E}_{\|}$, $\mathbf{B}'_{\|} = \mathbf{B}_{\|}$, $\mathbf{E}'_{\perp} = \gamma(\mathbf{E}_{\perp} + \mathbf{v} \times \mathbf{B}_{\perp})$, $\mathbf{B}'_{\perp} = \gamma(\mathbf{B}_{\perp} - \frac{\mathbf{v}\times\mathbf{E}_{\perp}}{c^2})$, where $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$).

Maxwell's field equations:

The required set of coupled equations governing $\mathbf{E}$ and $\mathbf{B}$ was "discovered" in 1864 by James Clerk Maxwell to be (first introduced in PHYS 212 in integral form and discussed throughout this course)

$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_o}$	Divergence eqn's	$\nabla \cdot \mathbf{B} = 0$
$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	Curl eqn's	$\nabla \times \mathbf{B} = \mu_o \mathbf{J} + \mu_o \epsilon_o \frac{\partial \mathbf{E}}{\partial t}$

where

$$\mu_o \equiv 4\pi \times 10^{-7} \, \frac{\mathbf{H}}{\mathbf{m}} \ \ \textbf{and} \ \ \epsilon_o = \frac{1}{\mu_o c^2} \approx \frac{1}{36\pi \times 10^9} \, \frac{\mathbf{F}}{\mathbf{m}}$$

in mksA units and

$$c = \frac{1}{\sqrt{\mu_o \epsilon_o}} \approx 3 \times 10^8 \, \frac{\mathbf{m}}{\mathbf{s}}$$

is the speed of light in free space. Furthermore $\rho = \rho(\mathbf{r}, t)$ refers to the net charge density and $\mathbf{J} = \mathbf{J}(\mathbf{r}, t)$ to the current density in the measurement frame, whereas $\nabla \cdot \mathbf{E}$ and $\nabla \times \mathbf{E}$ refer to the **divergence** and **curl** of vector field $\mathbf{E}$ generated by partial differentiation of the orthogonal components of $\mathbf{E}$ (concepts introduced in MATH 241 and reviewed in Lecture 4).

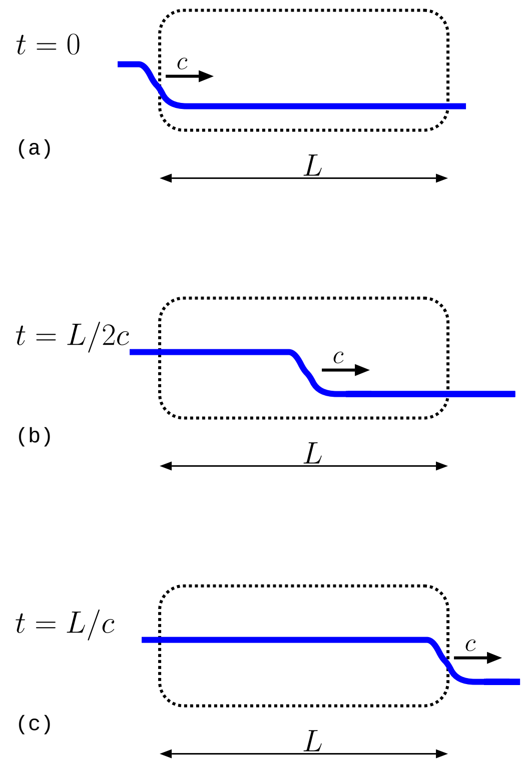
Solutions of Maxwell's equations — waves and static fields (AC/DC):

Maxwell's partial differential equations shown above, describing the coupled dynamics of electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ in response to space and time varying source fields ρ and $\mathbf{J}$, require an extended study to appreciate their full ramifications and predictions. All predictions of these equations have been experimentally verified and it has been found out that everything that is known and observed about electricity and magnetism can be explained in terms of these equations and their quantized forms.

One of their predictions, derived specifically in Lecture 18, is that they support *traveling wave solutions* of the form

$$\mathbf{E}(\mathbf{r},t) \propto \mathbf{B}(\mathbf{r},t) \propto \cos(2\pi f(t - \frac{z}{c})) \tag{6}$$

in regions where $\mathbf{J} = \rho = 0$. These are co-sinusoidal field perturbations having oscillation *frequencies* f, oscillation *periods* $T = \frac{1}{f}$, *wavelengths* $\lambda = \frac{c}{f}$, and they *travel* in 3D space with the speed of light c in free space. Since Maxwell's equations are linear, superpositions of co-sinusoidal waves with different wavelengths provide additional solutions — these can have arbitrary spatial variations and still travel at a fixed speed c. Any such field perturbation will travel across a region of size L during a time interval L/c as illustrated in the margin.



Another prediction of Maxwell's equations is that fields established by static — i.e., non-time-varying — charge and current densities $\rho = \rho(\mathbf{r})$ and $\mathbf{J} = \mathbf{J}(\mathbf{r})$ satisfy two separate sets of decoupled equations

Electrostatics		Magnetostatics
$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_o}$	Divergence eqn's	$\nabla \cdot \mathbf{B} = 0$
$\nabla \times \mathbf{E} = 0$	Curl eqn's	$\nabla \times \mathbf{B} = \mu_o \mathbf{J}$

shown in the left and right columns above — these were obtained by simply setting the terms $\partial\mathbf{E}/\partial t$ and $\partial\mathbf{B}/\partial t$ in the curl equations to zero. Independent "curl-free" static electric fields $\mathbf{E}(\mathbf{r})$ and "divergence-free" static magnetic fields $\mathbf{B}(\mathbf{r})$ satisfying these simplified equations are naturally far easier to determine than the coupled dynamic fields $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$ to be encountered in response to time-varying sources $\rho(\mathbf{r}, t)$ and $\mathbf{J}(\mathbf{r}, t)$.

Quasi-static fields:

Even though in practical cases of interest (in physics and engineering) time-varying sources are the "rule" and static sources an "exception", learning to solve the simplified set of *electrostatics* and *magnetostatics* equations turns out to be invaluable. The reason is, static solutions often provide accurate approximations — known as *quasi-static* approximation — for time-varying field problems involving slowly-varying sources $\rho(\mathbf{r}, t)$ and $\mathbf{J}(\mathbf{r}, t)$.

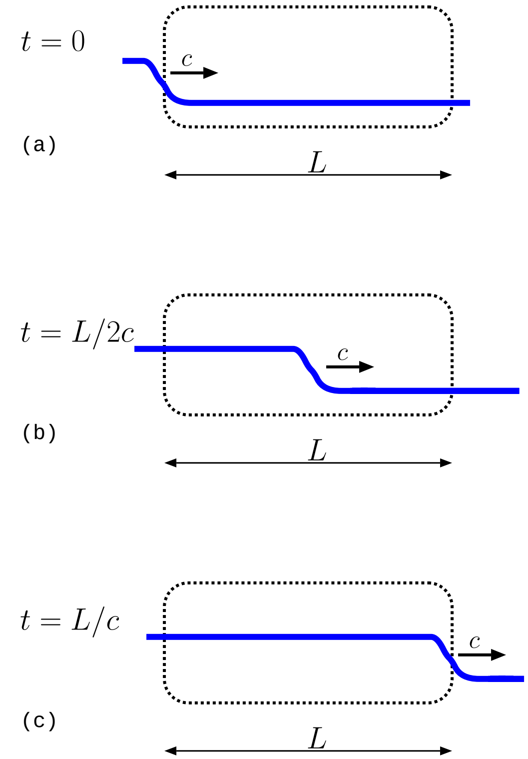
More specifically, if the source variation period T is much longer than the travel time L/c of field perturbations across a region of size L, that is, if

$$T \gg \frac{L}{c}, \tag{7}$$

then field calculations for the entire region can be done statically using the instantaneous (as opposed to retarded or previous) values of field sources ρ and $\mathbf{J}$. This is true because under the given condition source strengths will remain nearly constant over time intervals needed to communicate the new fields to the most distant corners of the region of interest. We can also re-state the same inequality (7) as

$$L \ll cT = \frac{c}{f} = \lambda \tag{8}$$

using the definition of wavelength λ introduced earlier. The indication is then, any system with a physical size L that is small in terms of wavelength λ of the applied field variations can be analyzed quasi-statically by starting from Maxwell's static equations.



Fields and circuits:

Lumped circuit analysis techniques introduced in ECE 110 and 210 constitute practical applications of the quasi-static approach suitable for "electrically small circuits" consisting of capacitors, inductors, and resistors and slowly varying AC sources. By contrast, the analysis of "electrically large circuits" with physical dimensions L approaching or exceeding λ requires taking a proper account of propagation time delays L/c in the system by developing a *distributed circuit* approach based on the full set of Maxwell's equations.

One practical application area where this need is most acute nowadays is in *chip* (integrated circuit) design and packaging suitable for high-speed computing[7]. While the physical dimensions of electronic chips and micro-circuits are generally very small, such elements can still be *electrically large* in the sense that $L \sim \lambda$ because of reduced wavelengths λ at high clock speeds $f = 1/T$. Thus, even the computer engineers (CompE's) amongst us need to understand and learn how to mitigate (and take advantage of) the ramifications of Maxwell's equations.

[7] E.g., Taflove, "Why study electromagnetics", *IEEE APM*, **44**, 132, 2002.

Details and study plan:

So much for the big picture about fields and waves encountered in nature and engineering systems and circuits. Working details of how fields and wave effects can be computed and characterized will be provided in the remaining parts of these notes.

Over the course of 39 lectures we will develop and study, in succession, the equations and applications of electrostatics (Lectures 1-11), magnetism (Lectures 12-15), and electromagnetics (Lectures 16-39) with a focus on time varying (quasi-static as well as wave-like) phenomena.

ECE 329:

We start by finding out how the equations of *electrostatics* arise from the familiar Coulomb's law (like charges repel, unlike charges attract) and the idea of field superpositions. We learn to solve electrostatic problems using the notion of *electrostatic potential* (voltage) and develop the notions of *polarization*, *conduction*, *charge continuity*, and *capacitance* in quasi-static settings of practical importance.

Next we learn how magnetic fields arise from charges in motion (a relative concept depending on the reference frame of the observer) and develop the governing laws of *magnetostatics* (also an extension of Coulomb's law seen from different reference frames). The *vector potential* is introduced for magnetic field calculations from prescribed current configurations, and notions of *magnetization* and *inductance* are subsequently developed and applied in quasi-static settings.

Just like time-varying electric fields imply time-varying charge densities (or *vice versa*) in electro-quasi-statics (EQS), time-varying currents imply time-varying magnetic flux in magneto-quasi-statics (MQS). We also learn that time-varying magnetic-flux is accompanied by time varying electric fields — a key finding of Faraday's called *induced field* with paradigm shifting ramifications and applications — and requires the modification of curl-free electric field condition into a dynamic equation known as Faraday's law.

Finally, the full set of Maxwell's equations is reached after adding a time-varying electric field term to the curl equation of magnetostatics. This change, first introduced by Maxwell in order to make sure that the governing equations of electricity and magnetism are consistent with *conservation of charge*, acknowledges the two-way coupling and feedback between electric and magnetic fields: time-varying magnetic fields induce time-varying electric fields — Faraday effect — and time-varying electric fields in turn induce time-varying magnetic fields (call it the "Maxwell effect") in order to sustain electromagnetic field variations in regions far away from charges and current loops — that is the way nature seems to work (and here we are to observe all that thanks to Maxwell effect allowing us to be here).

A study of wave solutions of Maxwell's equations follows, including *plane TEM waves* in free space, *linear and circular polarized* waves, waves in *conducting media*, *normal incidence* of waves on planar interfaces of homogeneous regions, *energy* and *momentum transfer*, *guided waves* in two-wire *transmission-line* (TL) systems, *transient response* on TL circuits, *resonant* *oscillations* in TL cavities, sinusoidal *steady-state* analysis of TL's and *distributed circuits*, *Smith Chart* applications, and finally *losses* in TL systems.

That is the full scope of the 39 lectures of ECE 329 — the course ends with an intensive study of distributed circuit concepts based on transmission lines, a study that complements the lumped circuit techniques examined and mastered in earlier courses.

ECE 329 is only the first half of our first-pass study of the fields and waves topics essential in electrical engineering education. Important topics such as radiation and antennas (generation details of electromagnetic waves by time-varying currents) and dispersion (frequency dependence of wave propagation speeds in material media) are barely mentioned or not at all in ECE 329. These constitute the main topics of the follow-on course, ECE 350.

ECE 350:

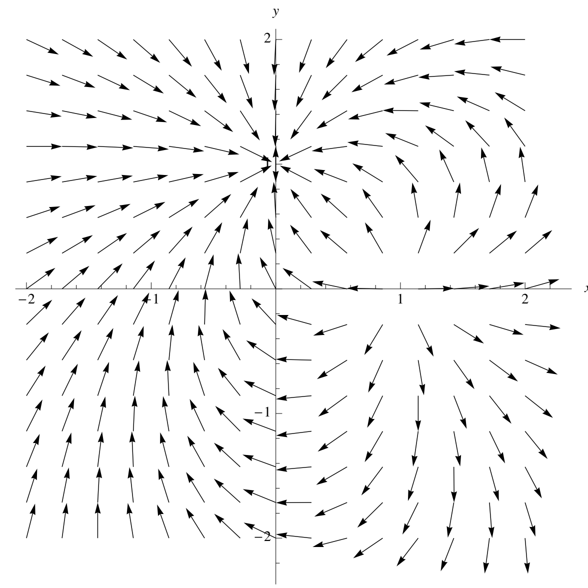
ECE 350 starts with the discussion of electromagnetic *radiation theory* and *transmission antennas*, continues with propagation and wave guidance effects (including *dispersion*, *phase and group velocities*, *Doppler shifts*, *oblique incidence*, *evanescence* and *tunneling* effects, *guided modes* in parallel-plate, rectangular, and dielectric slab *waveguides*), treats *cavity* fluctuations (including *resonant modes*, *blackbody radiation* in 3D cavities, *thermal noise*), and concludes with a discussion of *antenna reception* (including effective area, available power, link equations).

Beyond ECE 329 and 350:

Students having gone through ECE 329 and 350 will find themselves ready to encounter higher level courses in our curriculum focusing on different application areas and frequency regimes of the implications of Maxwell's equations. It is a life-long endeavor to master these relationships which have precipitated the scientific upheavals of the 20th century (relativity and quantum mechanics) and have remained intact and essential despite the upheavals unlike most aspects of classical physics. Our high speed electronics and communication networks and devices are intrinsically and fundamentally based on fields and wave concepts. Progress and innovation in these areas will require a deep understanding of fields and waves and how they interact with novel materials and structures.

Learn the basics and then go and invent the next thing!

1 Vector fields and Lorentz force



- Interactions between charged particles can be described and modeled[8] in terms of ***electric*** and ***magnetic*** **fields** just like gravity can be formulated in terms of ***gravitational fields*** of massive bodies.
  - In general, charge carrier dynamics and electromagnetic field variations[9] account for all electric and magnetic phenomena observed in nature and engineering applications.
- Electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ generated by charge carriers — *electrons* and *protons* at microscopic scales — permeate all space with proper time delays, and combine additively.
  - Consequently we associate with each location of space having Cartesian coordinates

$$(x, y, z) \equiv \mathbf{r}$$

a pair of time-dependent *vectors*

$$\mathbf{E}(\mathbf{r}, t) = (E_x(\mathbf{r}, t), E_y(\mathbf{r}, t), E_z(\mathbf{r}, t))$$

[8]Interactions can also be formulated in terms of *past locations* (i.e., trajectories) of charge carriers. Unless the charge carriers are stationary — i.e., their past and present locations are the same — this formulation becomes impractically complicated compared to field based descriptions.

[9]Time-varying fields can exist even in the absence of charge carriers as we will find out in this course — light propagation in vacuum is a familiar example of this.

and

$$\mathbf{B}(\mathbf{r},t) = (B_x(\mathbf{r},t), B_y(\mathbf{r},t), B_z(\mathbf{r},t))$$

that we refer to as **E** and **B** for brevity (dependence on position **r** and time t is *implied*).

- Field vectors **E** and **B** and electric charge and current densities ρ and **J** — describing the distribution and motions of charge carriers — are related by (i.e., satisfy) a coupled set of linear constraints known as **Maxwell's equations,** shown in the margin.
  - Maxwell's equations are expressed in terms of divergence and curl of field vectors — recall MATH 241 — or, equivalently, in terms of closed surface and line integrals of the fields enclosing arbitrary volumes V and surfaces S in 3D space, as you have first seen in PHYS 212.
    - Maxwell's equations were "discovered" as a consequence of experimental and theoretical studies led by 19th century scientists including Gauss, Ampere, Faraday, and Maxwell.

  They remain intact and essential despite the scientific upheavals (paradigm shifts) of 20th century: relativity and quantum physics[10].

Maxwell's equations:

$$\begin{aligned}\nabla\cdot\mathbf{E} &= \frac{\rho}{\epsilon_o}\\ \nabla\cdot\mathbf{B} &= 0\\ \nabla\times\mathbf{E} &= -\frac{\partial\mathbf{B}}{\partial t}\\ \nabla\times\mathbf{B} &= \mu_o\mathbf{J} + \mu_o\epsilon_o\frac{\partial\mathbf{E}}{\partial t}.\end{aligned}$$

such that

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B}),$$

with

$$\mu_o \equiv 4\pi\times 10^{-7}\,\frac{\mathbf{H}}{\mathbf{m}},$$

and

$$\epsilon_o = \frac{1}{\mu_o c^2} \approx \frac{1}{36\pi\times 10^9}\,\frac{\mathbf{F}}{\mathbf{m}},$$

in mksA units, where

$$c = \frac{1}{\sqrt{\mu_o\epsilon_o}} \approx 3\times 10^8\,\frac{\mathbf{m}}{\mathbf{s}}$$

is the speed of light in free space.

(In Gaussian-cgs units $\frac{\mathbf{B}}{c}$ is used in place of B above, while $\epsilon_o = \frac{1}{4\pi}$ and $\mu_o = \frac{1}{\epsilon_o c^2} = \frac{4\pi}{c^2}$.)

[10] Fields are are utilized in different ways in classical and quantum electrodynamics, but Maxwell's field equations remain the same under both paradigms. Relativity theory is an updated model of space and time relations developed to achieve consistency with the implications of Maxwell's equations.

Given the charge and current densities ρ and $\mathbf{J}$, Maxwell's equations can be *solved* for the fields $\mathbf{E}$ and $\mathbf{B}$.

- Field solutions $\mathbf{E}$ and $\mathbf{B}$ in turn determine how a "test charge" q with mass m, position $\mathbf{r}$, and velocity $\mathbf{v} \equiv \dot{\mathbf{r}} = \frac{d\mathbf{r}}{dt}$ *accelerates* in accordance with **Lorentz force**

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

  and Newton's 2nd law $\mathbf{F} = \frac{d}{dt}m\mathbf{v}$ (in classical electrodynamics). As such

  - **electric field E** at any location $\mathbf{r}$ is the vector force per stationary (i.e., $\mathbf{v} = 0$) unit charge (i.e., $q = 1$ C),
  - **magnetic field B** describes an additional force per unit charge which is experienced by charges in motion ($\mathbf{v} \neq 0$) in the reference frame — typically called the "lab frame" — where $\mathbf{E}$ and $\mathbf{v}$ are measured.

Since Lorentz force equation has the same form in all *inertial* reference frames[11] (like all laws of physics, including Maxwell's equations) while the charge velocity $\mathbf{v}$ is clearly frame-of-reference dependent, it follows that the values of fields $\mathbf{E}$ and $\mathbf{B}$ must also be dependent on the reference frame[12].

Lorentz force

Units in mksA system:

- q[=]C=sA,
- $\mathbf{E}$[=]N/C=V/m,
- $\mathbf{B}$[=]V.s/m^2=Wb/m^2=T,
- ρ[=]C/m^3,
- $\mathbf{J}$[=]A/m^2,

where C, N, V, Wb, and T are abbreviations for *Coulombs, Newtons*, *Volts*, *Webers*, and *Teslas*, respectively.

Charge q is quantized in units of $e = 1.602 \times 10^{-19}$ C, a relativistic invariant.

[11] Coordinate systems in which particles not subjected to any force — or, if general relativistic effects are to be retained, particles subjected to gravitational forces only — follow linearly varying trajectories.

[12] Given $\mathbf{E}$ and $\mathbf{B}$ measured in the lab, $\mathbf{E}'$ and $\mathbf{B}'$ measured by an observer moving through the lab with a constant velocity $\mathbf{v}$ are well approximated by $\mathbf{E}' \approx \mathbf{E} + \mathbf{v} \times \mathbf{B}$ and $\mathbf{B}' \approx \mathbf{B} - \frac{\mathbf{v}\times\mathbf{E}}{c^2}$ so long as $|\mathbf{v}| \ll c = 3 \times 10^8$ m/s, the *speed of light* in free space.

- Charge carrier positions $\mathbf{r}$, velocities $\dot{\mathbf{r}}$, and accelerations $\ddot{\mathbf{r}} = \frac{\mathbf{F}}{m}$, *as well as* forces $\mathbf{F}$, fields $\mathbf{E}$ and $\mathbf{B}$, and current density $\mathbf{J}$ are all described, in general, in terms of **3D vectors**.

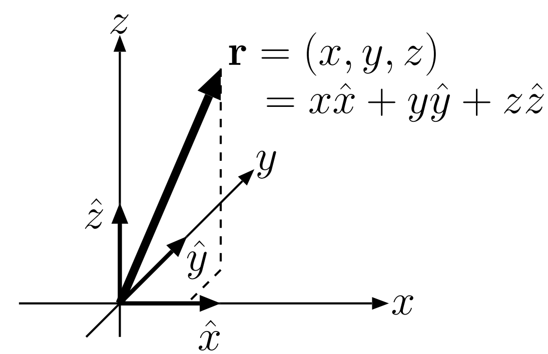
- In **Cartesian coordinates** such vectors and vector functions (of position $\mathbf{r}$ and/or time t) can be expressed in terms of mutually ***orthogonal* unit vectors** $\hat{x}$, $\hat{y}$, and $\hat{z}$ as in

  $$\mathbf{r} = (x, y, z) = x\hat{x}+y\hat{y}+z\hat{z} \quad \text{and} \quad \mathbf{E} = (E_x, E_y, E_z) = E_x\hat{x}+E_y\hat{y}+E_z\hat{z} \quad \text{etc.,}$$

  where

  - $|\mathbf{r}| \equiv \sqrt{x^2 + y^2 + z^2}$ and $|\mathbf{E}| \equiv \sqrt{E_x^2 + E_y^2 + E_z^2}$ etc., are **vector magnitudes**,
  - $\hat{r} \equiv \frac{\mathbf{r}}{|\mathbf{r}|}$ and $\hat{E} \equiv \frac{\mathbf{E}}{|\mathbf{E}|}$ etc., are associated **unit vectors**,
  - with ***dot products***
    - $\hat{r} \cdot \hat{r} = 1$, $\hat{E} \cdot \hat{E} = 1$, $\hat{x} \cdot \hat{x} = 1$, etc., but
    - $\hat{x} \cdot \hat{y} = \hat{x} \cdot \hat{z} = \hat{y} \cdot \hat{z} = 0$
  - and ***cross products***
    - $\hat{x} \times \hat{y} = \hat{z}$,
    - $\hat{y} \times \hat{z} = \hat{x}$,
    - $\hat{z} \times \hat{x} = \hat{y}$,

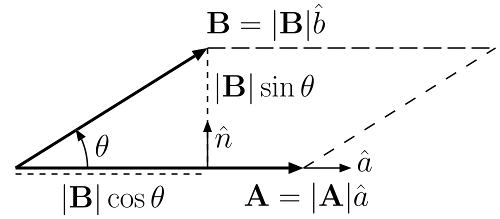
    adopting a **right-handed convention** (see the margin note in the next page).



UNIT VECTORS AND A POSITION VECTOR IN RIGHT-HANDED CARETESIAN COORDINATES

- Recall that
  - *Dot product* $\mathbf{A}\cdot\mathbf{B}$ is defined as $|\mathbf{A}|$ times $|\mathbf{B}|$ times the cosine of angle θ between $\mathbf{A}$ and $\mathbf{B}$.
    - Thus dot product is zero when angle θ is 90°, as in the case of $\hat{x}$ and $\hat{y}$, etc.
  - *Cross product* $\mathbf{A}\times\mathbf{B}$ is defined as a vector with a magnitude $|\mathbf{A}|$ times $|\mathbf{B}|$ times the sine of angle θ between $\mathbf{A}$ and $\mathbf{B}$ and a direction orthogonal to both $\mathbf{A}$ and $\mathbf{B}$ in a **right-handed** sense (see margin note) .
    - Thus cross product is zero when the vectors cross multiplied are collinear ($\theta = 0°$) or anti-linear ($\theta = 180°$).

Right handed convention: cross product vector points in the direction indicated by the thumb of your ***right hand*** when you rotate your fingers from vector $\mathbf{A}$ toward vector $\mathbf{B}$ through angle θ you decide to use.

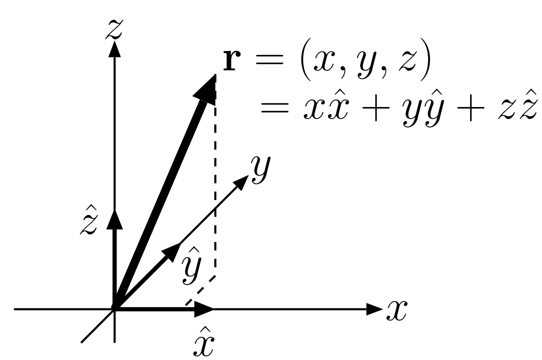


$\mathbf{A}\cdot\mathbf{B} = |\mathbf{A}||\mathbf{B}|\cos\theta$

DOT PRODUCT:product of projected vector lengths

$\mathbf{A}\times\mathbf{B} = |\mathbf{A}||\mathbf{B}|\sin\theta\hat{a}\times\hat{n}$

CROSS PRODUCT: right-handed perpendicular area vector of the parallelogram formed by co-planar vectors



UNIT VECTORS AND A POSITION VECTOR IN RIGHT-HANDED CARETESIAN COORDINATES

Example 1: Given the vectors $\mathbf{v} = (5, 10, 0)$ and $\mathbf{B} = (0, 0, 2)$ compute the cross and dot products $\mathbf{v}\times\mathbf{B}$ and $\mathbf{v}\cdot\mathbf{B}$.

Solution: Since we can also write $\mathbf{v} = 5\hat{x} + 10\hat{y}$ and $\mathbf{B} = 2\hat{z}$, it follows that

$$\mathbf{v}\times\mathbf{B} = (5\hat{x} + 10\hat{y})\times 2\hat{z} = 10\hat{x}\times\hat{z} + 20\hat{y}\times\hat{z} = -10\hat{y} + 20\hat{x}.$$

Alternatively, using the well known determinant method for cross products,

$$\mathbf{v}\times\mathbf{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 5 & 10 & 0 \\ 0 & 0 & 2 \end{vmatrix} = \hat{x}(10\cdot 2 - 0\cdot 0) - \hat{y}(5\cdot 2 - 0\cdot 0) + \hat{z}(5\cdot 0 - 10\cdot 0) = 20\hat{x} - 10\hat{y}.$$

Also, $\mathbf{v}\cdot\mathbf{B} = (5, 10, 0)\cdot(0, 0, 2) = 5\cdot 0 + 10\cdot 0 + 0\cdot 2 = 0.$

Example 2: A particle with charge $q = 1$ C passing through the origin $\mathbf{r} = (x, y, z) = 0$ of the lab frame is observed to accelerate with forces

$$\mathbf{F}_1 = 2\hat{x},\quad \mathbf{F}_2 = 2\hat{x} - 6\hat{z},\quad \mathbf{F}_3 = 2\hat{x} + 9\hat{y}\,\mathrm{N}$$

when the velocity of the particle is

$$\mathbf{v}_1 = 0,\quad \mathbf{v}_2 = 2\hat{y},\quad \mathbf{v}_3 = 3\hat{z}\,\frac{\mathrm{m}}{\mathrm{s}},$$

in turns. Use the Lorentz force equation

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B})$$

to determine the fields $\mathbf{E}$ and $\mathbf{B}$ at the origin.

Solution: Using the Lorentz force formula first with $\mathbf{F} = \mathbf{F}_1$ and $\mathbf{v} = \mathbf{v}_1$, we note that

$$2\hat{x} = (1)(\mathbf{E} + 0\times\mathbf{B}),$$

which implies that

$$\mathbf{E} = 2\hat{x}\,\frac{\mathrm{N}}{\mathrm{C}} = 2\hat{x}\,\frac{\mathrm{V}}{\mathrm{m}}.$$

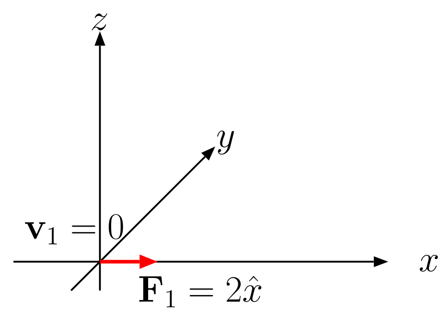
Next, we use

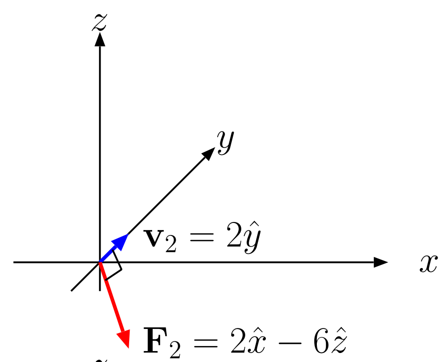
$$\mathbf{v}\times\mathbf{B} = \frac{\mathbf{F}}{q} - \mathbf{E} = \frac{\mathbf{F}}{q} - 2\hat{x}$$

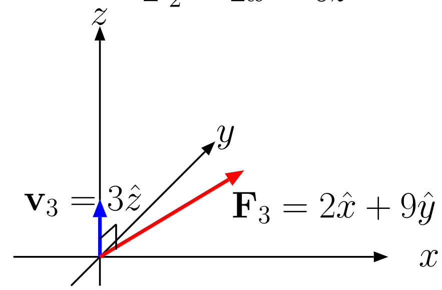
with $\mathbf{F}_2 = 2\hat{x} - 6\hat{z}$ and $\mathbf{v}_2 = 2\hat{y}$, as well as $\mathbf{E} = 2\hat{x}$ V/m, to obtain

$$2\hat{y}\times\mathbf{B} = -6\hat{z} \;\Rightarrow\; \hat{y}\times\mathbf{B} = -3\hat{z};$$

Having three non-colinear force measurements $\mathbf{F}_i$ corresponding to three distinct test particle velocities $\mathbf{v}_i$ is sufficient to determine the fields E and B at any location in space produced by distant sources as illustrated by this example.







likewise, with $\mathbf{F}_3 = 2\hat{x} + 9\hat{y}$ and $\mathbf{v}_3 = 3\hat{z}$,

$$3\hat{z} \times \mathbf{B} = 9\hat{y} \quad \Rightarrow \quad \hat{z} \times \mathbf{B} = 3\hat{y}.$$

Substitute $\mathbf{B} = B_x\hat{x} + B_y\hat{y} + B_z\hat{z}$ in above relations to obtain

$$\hat{y} \times (B_x\hat{x} + B_y\hat{y} + B_z\hat{z}) = -B_x\hat{z} + B_z\hat{x} = -3\hat{z}$$

and

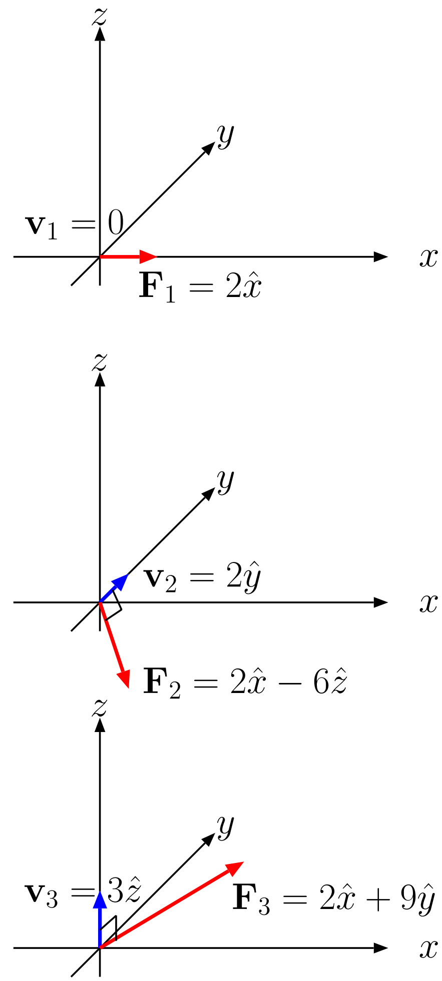
$$\hat{z} \times (B_x\hat{x} + B_y\hat{y} + B_z\hat{z}) = B_x\hat{y} - B_y\hat{x} = 3\hat{y}.$$

Matching the coefficients of $\hat{x}$, $\hat{y}$, and $\hat{z}$ in each of these relations we find that

$$B_x = 3\,\frac{\text{Wb}}{\text{m}^2}, \quad \text{and} \quad B_y = B_z = 0.$$

Hence, vector

$$\mathbf{B} = 3\hat{x}\,\frac{\text{Wb}}{\text{m}^2}.$$



- In your first homework you will be asked to do a sequence of vector exercises, including problems on volume, surface, and line integrals of vector or scalar functions of space (i.e., "fields"). These problems should be worked out with the help of your PHYS 212 and/or MATH 241 texts and notes.

  - **This course assumes a background of PHYS 212 and MATH 241 (on electromagnetic fields and vector calculus) as well**

as ECE 210 (lumped circuits and linear systems concepts including time- and frequency-domain approaches and phasors).

- The main objective of the course is to build up a **firm understanding of electromagnetic field concepts** introduced in PHYS 212, and to learn how to use **Maxwell's equations under static and time-varying conditions** associated with unguided (i.e., wireless) and guided (mainly transmission lines) electromagnetic waves. The study of **guided waves** is the key to extend the familiar **lumped-circuit** concepts into the realm of **distributed circuits**. This is the first half of a sequence of core electromagnetics courses in our curriculum, the second course being the 3-of-5 elective ECE 350.

  – Topical outline:

    1. **Static electric fields, potential, polarization, quasi-static applications** (10 lectures)
    2. **Static currents and magnetic fields** (3 lectures)
    3. **Time-varying fields and Maxwell's eqns** (4 lectures)
    4. **Plane wave solutions of Maxwell's eqns** (9 lectures)
    5. **Guided waves in transmission lines and distributed circuits** (13 lectures)

Prerequisites:
MATH 241
PHYS 212
ECE 210

Follow-on:
ECE 350